Course Notes
for
MS4303 Operations Research 1

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About the Course

- Lectures:
  - ⋆ Monday 14:00 S2–06
  - ⋆ Thursday 09:00 P1–033
- Tutorials: (starting in Week 2)
  - ⋆ (3B) Thursday 12:00 D1–050
  - ⋆ (3A) Thursday 16:00 C1–059
- Office Hours: (starting in Week 2)
  - ⋆ 15:00–16:00 Mondays,
  - ⋆ 16:00–17:00 Tuesdays,

  in B3-043.
- An attendance record will be kept.
- Assessment will be via a project and an end-of-semester written examination.

- The Course consists of topics from:
  - ⋆ Model building and the methods of operational research.
  - ⋆ Linear programming — graphical interpretation, Simplex Method and sensitivity analysis, duality and the Dual Simplex Method,
  - ⋆ Applications of Linear Programming — Transportation and assignment algorithms.
- The material is based closely on "Introduction to Linear Programming" by Ecker & Kupferschmid.
- A project will be assigned using a numerical software package (Matlab or Octave) to solve an LP problem.
- 30% of the marks for the course will be assigned to the project.
- 70% of the marks for the course will be assigned to the end-of-semester written examination.

1 Linear Programming Problems

Many practical problems in business & industry involve finding the best way to allocate scarce resources among competing activities. During and just after WW2, a technique called **Linear Programming** was developed to solve such problems using a systematic mathematical "algorithm" or procedure.

See `https://en.wikipedia.org/wiki/Linear_programming` for more historical background.

Linear Programming is one of the most important tools in Operations Research.

In this Chapter I'll introduce the topic using Examples.

1.1 An Introductory Example

I'll begin with a simple but typical example. All the numbers in this "toy" problem are small so that the arithmetic will not get in the way of understanding. A more realistic problem would deal with thousands of units of furniture and thousands of units of wood.

Example 1.1 (Chairs & Tables) *Every day, the T&C Corp has 12.5 units of wood in stock from which to make tables & chairs. Making a chair uses one unit of wood and a table uses two units. T&C's wholesale distributor will pay €15 for each chair & €20 for each table. For storage reasons the distributor cannot accept more than eight chairs in a daily transaction. Based on market research the distributor wants at least twice as many chairs as tables.*

(So the number of chairs is greater than or equal to twice the numbers of tables.) ⌣̈

How many chairs & tables should the company manufacture each day so as to maximise its revenue?

1.1.1 Formulation of the Problem

I will "formulate" this business decision as a Linear Programming problem. I'll drop the Euro € sign to avoid clutter.

I need only consider a single day's production as the decisions to be made are the same every day.

- Let x_1 and x_2 be the number of tables & chairs made **per day**.
  - ⋆ The variables x_1 and x_2 are called **decision variables** and a decision to manufacture x_1 tables and x_2 chairs **per day** is called a **production plan**.
  - ⋆ The income $z(\mathbf{x})$ that could be obtained from selling x_1 tables and x_2 chairs is $z = 20x_1 + 15x_2$.
  - ⋆ The distributor's requirements and the raw materials **constrain** how many tables & chairs can be made and sold **per day**.

- ⋆ The limit of eight chairs can be written as $x_2 \leq 8$ and the need for at least twice as many chairs as tables implies that $x_2 \geq 2x_1$.
- ⋆ Also x_1 and x_2 must be chosen so that $2x_1 + x_2 \leq 12.5$ due to the limited wood supply.
- ⋆ Finally, it doesn't make sense to make negative amounts of tables & chairs so I have $x_1 \geq 0$ and $x_2 \geq 0$.
- ⋆ These restrictions on x_1 and x_2 are called **constraints** and a production plan that simultaneously satisfies all the constraints is called **feasible**.

- Of course I could have labelled the number of chairs as x_1 and the number of tables as x_2!
- How would the diagram in Fig. 1 change?

- Each feasible **daily** production plan is a feasible choice of x_1 and x_2.
  - ⋆ Production plans for the T&C Corp problem are shown in Figure 1.
  - ⋆ Points representing feasible production plans are called **feasible points**.
  - ⋆ The set of all feasible points is called the **feasible set** or **feasible region** — shaded in blue in the Figure and labelled by $\mathcal{F}$.
  - ⋆ All the points in $\mathcal{F}$ satisfy all the constraints above.
  - ⋆ For example, picking $x_1 = 2$ and $x_2 = 7$ (the **red** dot in Fig 1) satisfies all the constraints and yields a revenue of $z = €145$.

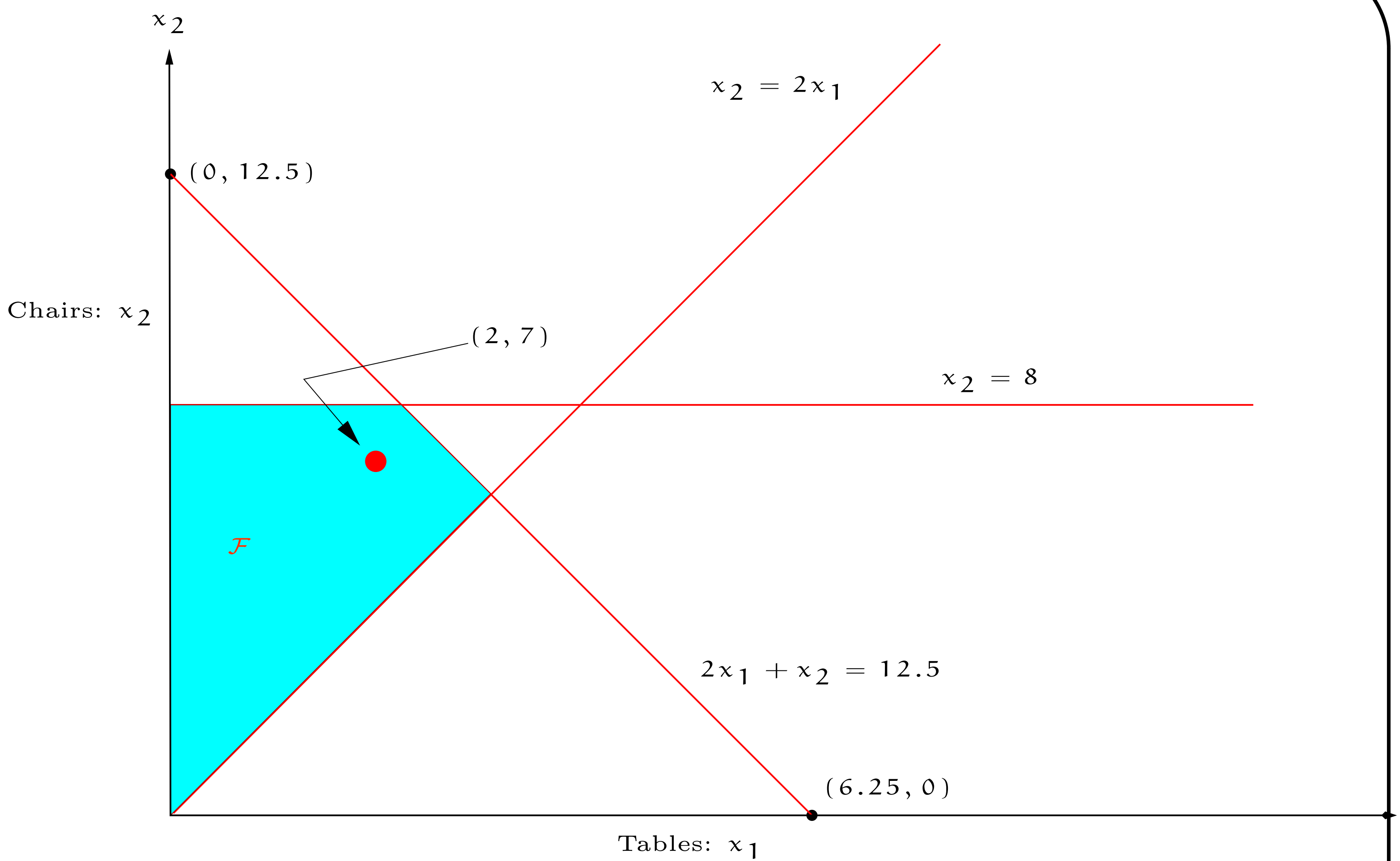

Figure 1: Feasible Set for the T&C Corp Problem

- Of course, many other combinations (points) (x_1, x_2) yield the same revenue — namely all the points on the **line** $20x_1 + 15x_2 = 145$.
  - ⋆ Other combinations of x_1 and x_2 (points (x_1, x_2)) can be found that also satisfy all the constraints and have **larger** revenues $z = 20x_1 + 15x_2$.
  - ⋆ The company T&C Corp needs to find the point (x_1, x_2) (production plan) that makes $z(\mathbf{x})$ as big as possible.
  - ⋆ The function $z = 20x_1 + 15x_2$ is called the **objective function** for the problem.

- The line corresponding to the equation $z = 20x_1 + 15x_2 = 145$ is called a **contour** of the objective function.
  - ⋆ This line is shown in Figure 2, passing through the point $(2, 7)$.
  - ⋆ Some other objective function contours, corresponding to different values of $z(\mathbf{x})$, are also shown.

- From Figure 2 you can see that the contours of the objective function are parallel lines.
  - ⋆ This is **always** true for Linear Programs (LP's).
  - ⋆ Why?
  - ⋆ The value of the objective function $z(\mathbf{x})$ increases as the contour lines move up and right.
  - ⋆ Corresponding to x_1 and/or x_2 increasing.
  - ⋆ For a point (x_1, x_2) to be feasible, it must lie in (or on the edge of) the feasible region $\mathcal{F}$.
  - ⋆ As none of the points on the contour line for $z = 200$ are in the feasible set $\mathcal{F}$, a revenue of €200 cannot be achieved by **any** feasible production plan.

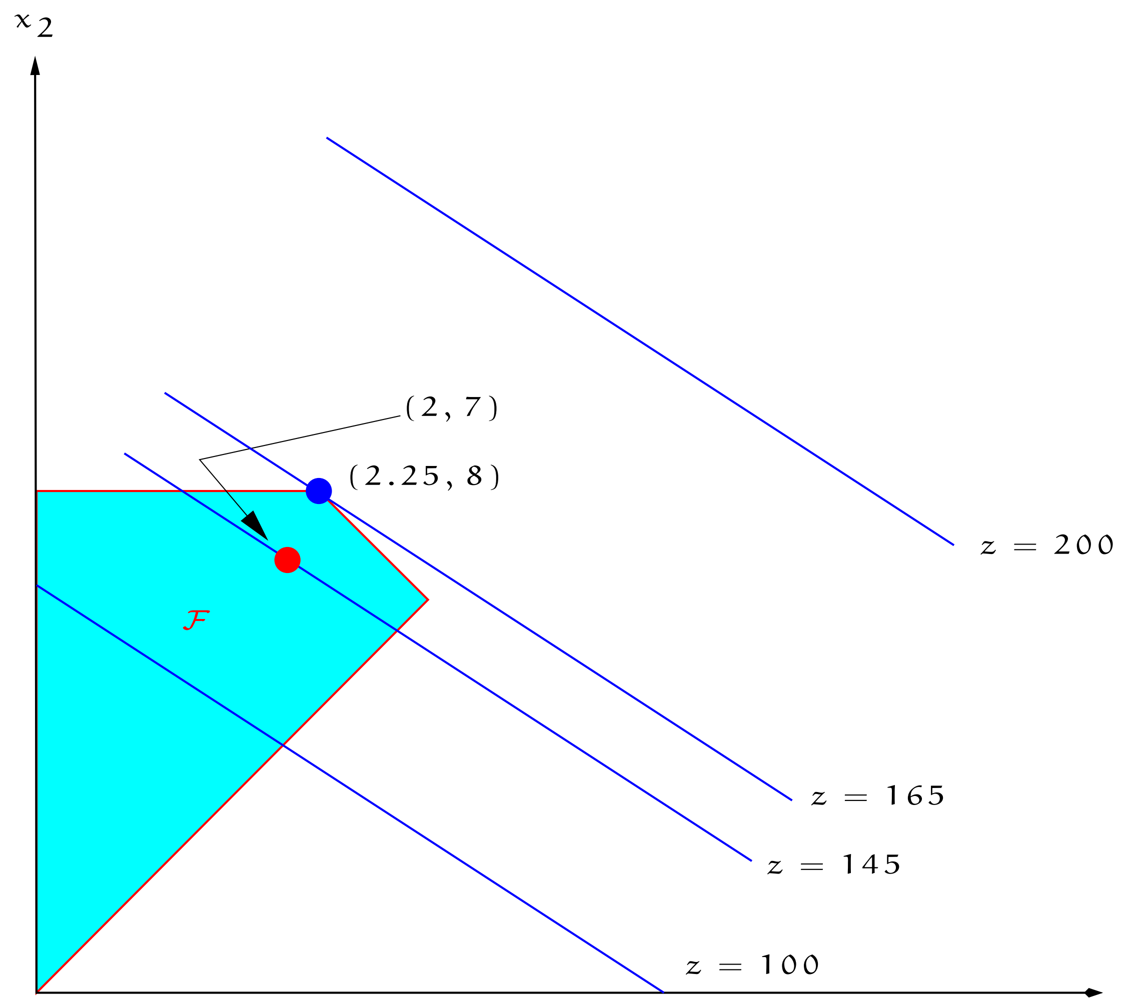

Figure 2: Objective Function Contours for the T&C Corp Problem

- Of all the objective function contours that intersect the feasible region $\mathcal{F}$ at at least one point, the one with the highest $z(\mathbf{x})$–value is the line $z = 165$.
  - ⋆ This line has only one feasible point, the **blue** dot in Figure 2 namely $x_1 = 2.25$ and $x_2 = 8$.
  - ⋆ This is the best possible production plan for T&C.
- The point (or points) that give the largest possible objective fuction ($z(\mathbf{x})$) value while satisfying all the constraints is called the **optimal point** for the Linear Programming problem.
- The term **solution** is often used for the **optimal point**.
- Confusingly, (especially in older books) the term **solution** is sometimes (not in these Notes) used for feasible points!
- The corresponding objective function value is called the **optimal value**.

- The process of finding the maximum or minimum value of an objective function (usually subject to constraints) is called **optimisation**.
- So Linear Programming is **linear optimisation**.

1.1.2 Does the Solution Need to be an Integer?

- The optimal point that I found for the T&C Corp problem was $x_1 = 2.25$ and $x_2 = 8$.
- But how can T&C Corp produce $2\frac{1}{4}$ tables?
- And would the distributor want a $\frac{1}{4}$ of a table?
- If I was really interested in finding the solution to such a small problem, I would need to find an integer solution — see Figure 3.
- The integer points are shown as black dots.
- The nearest integer point to $(2\frac{1}{4}, 8)$ is the **magenta** dot in Fig. 3 $(2, 8)$ — this is the **integer optimal point**.
- The corresponding revenue is $z = €160$.

- For most practical problems (lots of constraints, lots of decision variables) the fact that the optimal point is not an exact integer doesn't matter.
- The variables may represent quantities that don't have to be integer (litres of liquids, units of electricity).
- Or the amounts may be so large that the non-integer solution can be safely rounded off to give an integer solution.
- **But** sometimes rounding doesn't work.
- If I really need to find integer optimal points, special methods are available — "Integer Programming" methods.
- These methods will be explained in OR 2 in Fourth Year.
- From now on in this course I will accept solutions irrespective of whether or not they are integer.

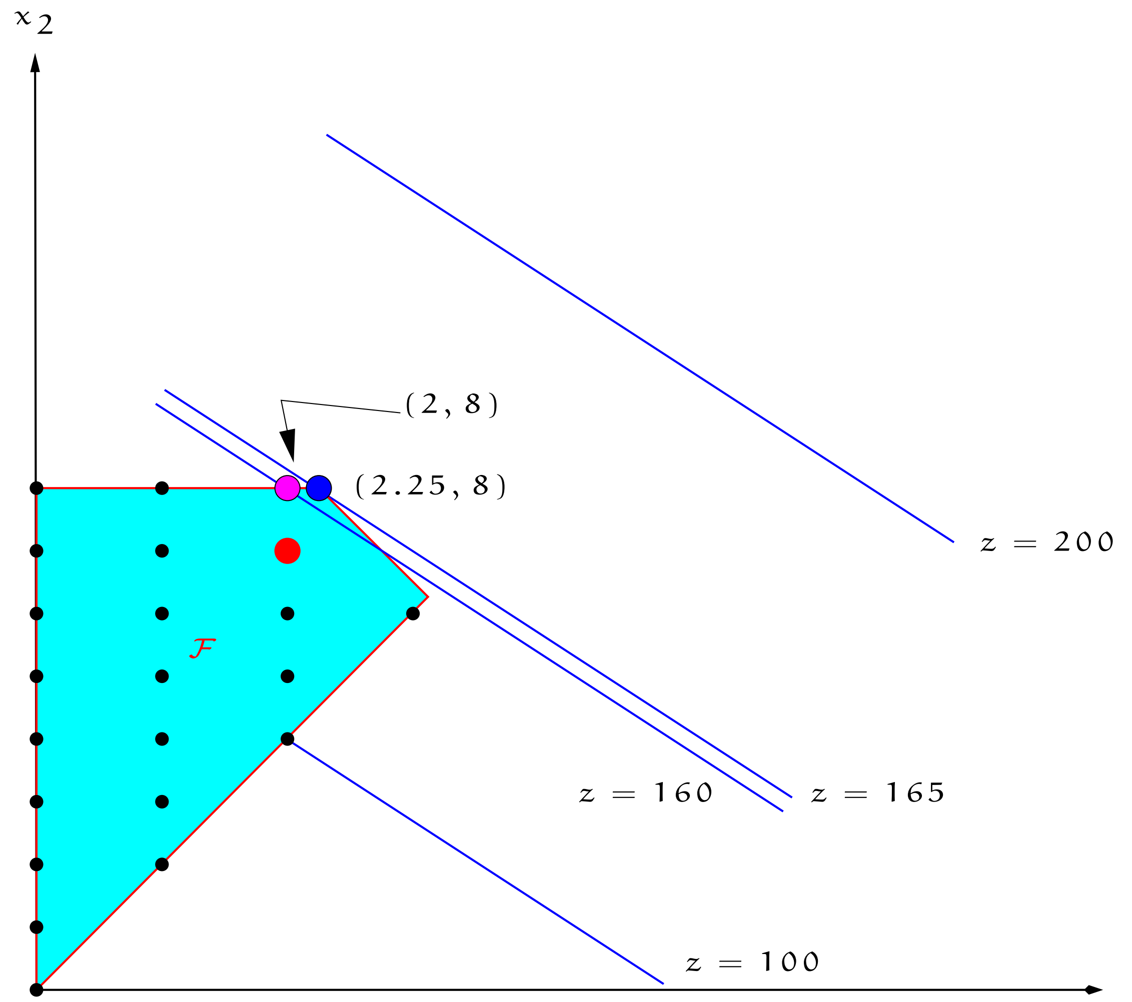

Figure 3: Integer Feasible Set for the T&C Corp Problem

Stopped here 15:00, Monday Week 1

1.1.3 Sensitivity of the Optimal Point

- Often the input data (problem specs) for a LP are only known approximately.
- So an important question is: "how sensitive is the optimal point to small changes in the data?".
- For example, suppose that the distributor in the T&C Corp problem is uncertain about the price (s)he can get when (s)he sells the furniture on to the end user (retail customer).
- Suppose that as a result the distributor may be prepared to pay as little as €11 or as much as €19 per chair — the standard price is €15.
- The obvious question now is:
  - ⋆ Does the optimal production plan change?
  - ⋆ Should I change the amounts of x_1 and x_2 to be produced?

Exercise 1.1 *Check using graphical methods that, even though the objective function contours change (and so the optimal value* $z(\mathbf{x})$ *changes), the optimal point* $x_1 = 2\frac{1}{4}$, $x_2 = 8$ *is not affected by varying the wholesale price of chairs within the range €11–€19.*

Exercise 1.2 *What is the maximum variation (up or down) in the wholesale price of chairs that keeps the optimal point* $x_1 = 2\frac{1}{4}$, $x_2 = 8$ *unchanged?*

- So the T&C Corp problem is not very sensitive to the wholesale price of chairs.
- Just as the graphical method demonstrated above is only practical for toy problems with two decision variables, re-solving the problem from scratch is also not practical for real problems.
- Later in the course I will explain some practical methods for sensitivity analysis.

1.1.4 Algebraic Statement of Linear Programming Problems

- The geometric picture and interpretation give important insights.
- But as mentioned above, for more than two decision variables graphical methods are generally useless.
- Real LPs have thousands of decision variables and constraints.
- Solution methods need to work on the **algebraic** statement of the problem, not on pictures.

- An algebraic description of the T&C Corp problem is:

$$
\begin{aligned}
\max z = 20x_1 + 15x_2 & \qquad \text{Objective function} \\
\text{subject to} & \\
x_2 \leq 8 & \qquad \text{Constraint 1} \\
2x_1 - x_2 \leq 0 & \qquad \text{Constraint 2} \\
2x_1 + x_2 \leq 12.5 & \qquad \text{Constraint 3} \\
x_1, x_2 \geq 0. & \qquad \text{Non-negativity Constraints}
\end{aligned}
$$

- Such an algebraic formulation of a problem is called a **linear program** (**LP**).
- Any LP problem can be stated algebraically in many different ways — all equivalent.
- Every LP has the following general form:

max / min **Linear Objective Function**

subject to

Linear Equality Constraints

and/or

Linear Inequality Constraints

- In the next Section I'll give some more example problems and show how they can be formulated as LPs.

1.2 More LP Formulation Examples

I'll list a couple of Examples, then I'll examine each in turn.

Example 1.2 (Pet Food) *The K9 Pet Food Company manufactures two types of dog food: Better and Worse.*

- *Each pack of Better contains 2 kg of cereal and 3 kg of meat.*
- *Each pack of Worse contains 3 kg of cereal and 1.5 kg of meat.*
- *K9 believes it can sell as much of each dog food as it can make (there are a lot of dogs out there).*
- *Better sell for €2.80 per pack and Worse sell for €2.00 per pack.*

K9's production is limited in several ways.

- *First, K9 can buy only up to 400,000 kg of cereal each month at €0.20 per kg.*
- *It can buy only up to 300,000 kg of meat per month at €0.50 per kg.*
- *In addition, a special piece of machinery is required to make Better and this machine has a capacity of 90,000 packs per month.*
- *The variable cost of blending and packing the dog food is €0.25 per pack for Better and €0.20 per pack for Worse.*

This information is given in the table on the next Slide.

	Better	*Worse*
Sales price per pack	*€2.80*	*€2.00*
Raw materials per pack		
Cereal	*2.0 kg*	*3.0 kg*
Meat	*3.0 kg*	*1.5 kg*
Variable costs —blending and packing	*€0.25 pack*	*€0.20 pack*
Resources		
Production capacity for Better	*90,000 packs per month*	
Cereal available per month	*400,000 kg*	
Meat available per month	*300,000 kg*	

Suppose you are the manager of the K9 Pet Food Company. Your salary is based on Company profits, so you try to maximise its profit. How should you operate the Company to maximize its profit and your salary?

(A solution will appear here.)

Stopped here *10:00, Thursday Week 1*

Solution:

[The Decision Variables] *We first identify those things over which we have control: the decision variables.*

- *In this problem we have direct control over two quantities: the number of packs of Better to make each month and the number of packs of Worse to make each month.*
- *Within the model these two quantities appear repeatedly, so we represent them in a simple fashion.*
- *We designate these variables by the symbols* x_1 *and* x_2 *(in alphabetical order).*
  - ⋆ x_1 = *number of packs of Worse to make per month.*
  - ⋆ x_2 = *number of packs of Better to make per month*

Note that the amount of meat used each month and the amount of cereal used each month are not good choices for the variables.

- *First, we control these only indirectly through our choice of* x_1 *and* x_2.
- *More important, using these as variables could lead to ambiguous production plans.*
- *Determining how much cereal and meat to use in production does not tell us how to use it — how much of each type of dog food to make.*
- *In contrast, after determining the values for* x_1 *and* x_2, *we know what to produce and how much meat and cereal are needed.*

[Intermediate Variables] • *It is useful to designate* c *and* m *as the volume of cereal & meat to be purchased respectively.*

- *For any choice of* x_1 *and* x_2 *to be produced I must purchase* $3x_1 + 2x_2$ *units of cereal so* $c = 3x_1 + 2x_2$.
- *Similarly for any choice of* x_1 *and* x_2 *to be produced I must purchase* $1.5x_1 + 3x_2$ *units of meat so* $m = 1.5x_1 + 3x_2$.

[Constraints] *If we want to make z as large as possible, why not increase* x_1 *and* x_2 *to infinity and earn an infinite profit?*

- *We cannot do this because there are limits on the availability of cereal and meat (*c *and* m*) and on the production capacity for Better.*
  - ⋆ *In reality, there is also a limit on demand but we ignore that here for simplicity.*
- *We want to maximize* z*, but subject to satisfying the stated constraints.*
- *To solve the problem, we express these constraints as mathematical equalities or inequalities.*

- *Begin with the availability of cereal constraint:* $c \leq 400,000$.
- *Substituting for* c *in terms of* x_1 *and* x_2 *I can write the constraint as* $3x_1 + 2x_2 \leq 400,000$.
- *Similarly, the availability of meat constraint:* $m \leq 300,000$.
- *So the restriction on the availability of meat can be expressed in terms of* x_1 *and* x_2 *as* $1.5x_1 + 3x_2 \leq 300,000$.
- *In addition to these constraints, the number of packs of Better produced each month can not exceed 90,000; that is,* $x_2 \leq 90,000$.
- *Finally, negative production levels do not make sense, so we require that* $x_1 \geq 0$ *and* $x_2 \geq 0$.

[Objective Function] *Any* **feasible** *pair of numerical values for the variables x_1 and x_2 is a production plan.*

- *For example, $x_1 = 20,000$ and $x_2 = 10,000$ means we make 20,000 packs of Worse and 10,000 packs of Better each month.*
- *But how do we know whether this is a good production plan?*
- *We need to specify a criterion for evaluation — an objective function.*
- *The most appropriate objective function is to maximize monthly profit.*
  - ⋆ *Fixed costs are ignored because any plan that maximizes revenue minus variable costs maximizes profit as well.*

The profit earned by K9 is a direct function of the amount of each dog food made and sold, the decision variables.

- *Monthly profit, designated as z, is written as follows:*
  $z =$ *(profit per pack of Worse)* $\times$
  (number of packs of Worse made and sold monthly) $+$
  (profit per pack of Better) $\times$
  (number of packs of Better made and sold monthly).
- *The profit per pack for each dog food is computed as follows:*

	Worse x_1	Better x_2
Selling price	2.00	2.80
Minus		
Meat	0.75	1.50
Cereal	0.60	0.40
Blending	0.20	0.25
Profit per pack	0.45	0.65

- *We write the month's profit as* $z = 0.45x_1 + 0.65x_2$.

Putting all these together gives the following LP (constrained optimization model).

Maximize $z = 0.45x_1 + 0.65x_2$

Subject to:

$$\begin{aligned} 3x_1 + 2x_2 &\leq 400,000 \\ 1.5x_1 + 3x_2 &\leq 300,000 \\ x_2 &\leq 90,000 \\ x_1, x_2 &\geq 0. \end{aligned}$$

- *This type of model is called a linear programming model or a linear program because the objective function is linear and the functions in all the constraints are linear.*
- *The optimum solution for the K9 problem is* $x_1 = 100,000$ *and* $x_2 = 50,000$.
- *Check that the constraints are all satisfied.*
- *Substitute the values of* x_1 *and* x_2 *into* z *and check that this yields a profit* $z =$ *€77,500.*
- *That is, K9 should make 100,000 packs of Worse and 50,000 packs of Better and each month. It will earn a monthly profit of €77,500.*
- *But — how do I know this is the best solution?* ☺
- *And how did I find it?*

Example 1.3 (Dear Beer Co.) *The Dear Beer Co. brews four different beers; Ale, Lager, Premium & Stout.*

- *These beers are brewed using the following resources (ingredients): hops, malt, water & yeast.*
- *Dear Beer Co. have a free water supply (for the moment) so the availability of the other three ingredients restrict production capacity.*
- *The table below gives the number of kilogrammes of each input needed to produce 1 unit (8 litres, approx 2 gallons) of product together with the revenue received per unit.*

	Ale	*Lager*	*Premium*	*Stout*	*Stock*
Malt	*1*	*1*	*0*	*3*	*50kg*
Hops	*2*	*1*	*2*	*1*	*150kg*
Yeast	*1*	*1*	*1*	*4*	*80kg*
Revenue/unit	*€6*	*€5*	*€3*	*€7*	

- *Formulate the problem as an LP.*

(A solution will appear here.)

Solution:

[Decision Variables] x_1, x_2, x_3 and x_4 corresponding to units of Ale, Lager, Premium & Stout to be produced.

[Intermediate Variables] m, h & y.

$$\begin{aligned} m &= x_1 + x_2 + 0x_3 + 3x_4 \\ h &= 2x_1 + x_2 + 2x_3 + x_4 \\ y &= x_1 + x_2 + x_3 + x_4 \end{aligned}$$

[Constraints]

$$\begin{aligned} m \le 50 &\equiv x_1 + x_2 + 0x_3 + 3x_4 \le 50 \\ h \le 150 &\equiv 2x_1 + x_2 + 2x_3 + x_4 \le 150 \\ y \le 80 &\equiv x_1 + x_2 + x_3 + x_4 \le 80 \end{aligned}$$

together with $x_i \ge 0$, $i = 1, \ldots, 4$.

[Objective Function] Simply $z = 6x_1 + 5x_2 + 3x_3 + 7x_4$.

So I must solve the following LP:

Maximize $z = 6x_1 + 5x_2 + 3x_3 + 7x_4$

Subject to:

$$\begin{aligned} x_1 + x_2 + 0x_3 + 3x_4 &\leq 50 \\ 2x_1 + x_2 + 2x_3 + x_4 &\leq 150 \\ x_1 + x_2 + x_3 + x_4 &\leq 80 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

- Again this is a linear programming model or a LP because the objective function is linear and the functions in all the constraints are linear.
- The optimum solution for the Dear Beer problem is $x_1 = 40$, $x_2 = 10$, $x_3 = 30$ and $x_4 = 0$.
- So produce 40 units of Ale, 10 units of Lager and 30 units of Premium.
- No Stout to be made!
- Check that the constraints are all satisfied.
- Substitute the values of $x_1, \ldots, x_4$ into z and check that this yields a profit $z = €380$.
- But — how do I know this is the best solution? $\ddot\smile$
- And how did I find it?

Example 1.4 (Oil Refinery) *An oil refinery can blend three grades of crude oil (A, B & C) to produce diesel & petrol.*

- *Two possible blending processes are available, the older & the newer.*
- *For each production run,*
  - ⋆ *the older process uses 5 units of crude A, 7 units of crude B & 2 units of crude C to produce 9 units of diesel and 7 units of petrol.*
  - ⋆ *the newer process uses 3 units of crude A, 9 units of crude B & 4 units of crude C to produce 5 units of diesel and 9 units of petrol..*
- *To meet demand, the refinery must produce at least 500 units of diesel & at least 300 units of petrol each month.*

- *The refinery has available 1500 units of crude A, 1900 units of crude B & 1000 units of crude C.*
- *For each unit of diesel produced, the refinery receives €6 and for each unit of petrol produced, the refinery receives €9.*
- *Formulate the problem as an LP.*

(A solution will appear here.)

Solution: x_1 and x_2 are the dv's, # runs of old & new process respectively.

The intermediate variables are:

$$\begin{aligned} A &= 5x_1 + 3x_2 \\ B &= 7x_1 + 9x_2 \\ C &= 2x_1 + 4x_2 \\ D &= 9x_1 + 5x_2 \\ P &= 7x_1 + 9x_2 \end{aligned}$$

The constraints are:

$$\begin{aligned} A &\leq 1500 \\ B &\leq 1900 \\ C &\leq 1000 \\ D &\geq 500 \\ P &\geq 300 \end{aligned}$$

The objective function is $z = 6D + 9P$

In terms of x_1 & x_2:

$$\begin{array}{rrrrrrrcr}
5x_1+ & 3x_2+ & 1x_3+ & 0x_4+ & 0x_5+ & 0x_6+ & 0x_7 & = & 1500\\
7x_1+ & 9x_2+ & 0x_3+ & 1x_4+ & 0x_5+ & 0x_6+ & 0x_7 & = & 1900\\
2x_1+ & 4x_2+ & 0x_3+ & 0x_4+ & 1x_5+ & 0x_6+ & 0x_7 & = & 1000\\
-9x_1 & -5x_2+ & 0x_3+ & 0x_4+ & 0x_5+ & 1x_6+ & 0x_7 & = & -500\\
-7x_1 & -9x_2+ & 0x_3+ & 0x_4+ & 0x_5+ & 0x_6+ & 1x_7 & = & -300
\end{array}$$

The problem is a max one, so minimise $z = -(117x_1 + 111x_2)$.

See `http://jkcray.maths.ul.ie/ms4303/fuel.m` for a simple matlab script m-file that creates a starting tableau for the problem.

It should look like this:

0	−117	−111	0	0	0	0	0
1500	5	3	1	0	0	0	0
1900	7	9	0	1	0	0	0
1000	2	4	0	0	1	0	0
−500	−9	−5	0	0	0	1	0
−300	−7	−9	0	0	0	0	1

Try to pivot (using `http://jkcray.maths.ul.ie/ms4303/Pivot.m`) to this optimal tableau:

$31,757_{1/7}$	0	$39_{3/7}$	0	$16_{5/7}$	0	0	0
$142_{6/7}$	0	$-3_{3/7}$	1	$_{-5/7}$	0	0	0
$1,942_{6/7}$	0	$6_{4/7}$	0	$1_{2/7}$	0	1	0
$457_{1/7}$	0	$1_{3/7}$	0	$_{-2/7}$	1	0	0
$271_{3/7}$	1	$1_{2/7}$	0	$_{1/7}$	0	0	0
$1,600$	0	0	0	1	0	0	1

I've written the decimal parts as exact fractions for readability.

Here is the optimal tableau where the fractions are written as ratios:

222300/7	0	276/7	0	117/7	0	0	0
1000/7	0	−24/7	1	−5/7	0	0	0
13600/7	0	46/7	0	9/7	0	1	0
3200/7	0	10/7	0	−2/7	1	0	0
1900/7	1	9/7	0	1/7	0	0	0
1600	0	0	0	1	0	0	1

Which do you prefer?

Example 1.5 (Warehouse) *A large apple storage warehouse has the capacity to store 200 tonnes of apples for an indefinite period.*

- *There are currently 80 tonnes of apples in the warehouse.*
- *The Sales Manager has produced a forecast for the per-tonne prices for apples at the next five monthly wholesale apple sales — listed in the Table below.*

Month	*Price*
1	*120*
2	*100*
3	*150*
4	*180*
5	*130*

- *There is also a storage cost of €5/tonne/month for apples left in the warehouse after each sale.*
- *As the (forecast) price of apples varies from month to month, it is possible to make a profit by buying cheap & selling dear.*
- *For example, the warehouse could buy apples in month 2 at €100 per tonne, store them till month 4 & then sell them at €180 per tonne.*
- *What should the buying & selling programs be to maximise the total profit made at the next five monthly apple sales?*
- *Formulate the problem as an LP.*

(A solution will appear here.)

Solution: *Let's suppose that buying & selling are done at the start of each month. The DV's are* b_i = *tonnes bought &* s_i = *tonnes sold each month,* $i = 1, \ldots, 5$. *The IV's are the amount in storage for month* i; k_i, *say ("k for keep") which is given by:*

$$
\begin{aligned}
k_1 &= 80 + b_1 - s_1 \\
k_2 &= k_1 + b_2 - s_2 \\
k_3 &= k_2 + b_3 - s_3 \\
k_4 &= k_3 + b_4 - s_4 \\
k_5 &= k_4 + b_5 - s_5
\end{aligned}
$$

Constraints: $b_i \geq 0$, $s_i \geq 0$, $0 \leq k_i \leq 200$ *for* $i = 1, \ldots, 5$. *It is reasonable to add the constraint* $k_5 = 0$ *as it doesn't make sense to store apples in the last month rather than sell them.*

Use the term p_i, $i = 1, \ldots, 5$ *for the buy/sell price for each month.*

Then (see comment on prev. slide about k_5*)*

$$z = (s_1 - b_1)p_1 + (s_2 - b_2)p_2 + (s_3 - b_3)p_3 + (s_4 - b_4)p_4 + (s_5 - b_5)p_5 \\ - 5(k_1 + k_2 + k_3 + k_4 + k_5).$$

It would also be reasonable to add the constraint $b_5 = 0$ *but an optimal solution would set* b_5 *to zero anyway (why?) so no need.*

Example 1.6 (Chickens and Eggs) *The owner of a small chicken farm must determine a laying and hatching program for 100 hens.*

- *The number of hens doesn't change over the period examined.*
- *At the start there are 100 eggs in the henhouse.*
- *Each hen can be used to hatch existing eggs or to lay new ones.*
- *In each 10-day period, a hen can either hatch 4 existing eggs or lay 12 new eggs.*
- *Chicks that are hatched are sold immediately for 60 cent each.*
- *Every 30 days an egg dealer will pay 10 cent each for the eggs accumulated at that point.*
- *Eggs not being hatched in one period can be kept in an incubator room for hatching in a later period.*

- *The problem is to determine how many hens should be hatching and how many laying in each of the next three 10-day periods so that total revenue is maximised.*

- *Formulate the problem as an LP — tricky but do–able...*

- *The problem, once formulated, is easily solved with the Simplex Method, to be explained below.*

- *The answer, once found, is easy to understand in hindsight as we will see..*

  Stopped here *15:00, Monday Week 2*

  (A solution will appear here.)

  Solution: *Let* h_1, h_2, h_3 *be number of hens hatching in each of the 3 periods. Obviously, the number laying in each period is* $l_i = (100 - h_i)$, $i = 1, 2, 3$.

Then,

$$
\begin{aligned}
NE_0 &= 100; \\
NE_1 &= NE_0 - 4h_1 + 12l_1 \\
&= NE_0 - 4h_1 + 12(100 - h_1) \\
&= NE_0 + 1200 - 16h_1 = 1300 - 16h_1; \\
NC_1 &= 4h_1; \\
NE_2 &= NE_1 - 4h_2 + 12l_2 \\
&= NE_1 - 4h_2 + 12(100 - h_2) = 2500 - 16(h_1 + h_2); \\
NC_2 &= 4h_2; \\
NE_3 &= NE_2 - 4h_3 + 12l_3 \\
&= NE_2 - 4h_3 + 12(100 - h_3) = 3700 - 16(h_1 + h_2 + h_3); \\
NC_3 &= 4h_3.
\end{aligned}
$$

Constraints:

$h_i \geq 0, h_i \leq 100,$ ($\textit{equiv. to } l_i \geq 0$), $NE_i \geq 0, i = 1, 2, 3.$

Revenue

$z = 10 * NE_3 + 60 * (NC_1 + NC_2 + NC_3) = 37000 + 80(h_1 + h_2 + h_3).$

Init tableau: (NB $\max z \equiv \min -z$*),*
$-z = -37000 - 80(h_1 + h_2 + h_3)$*.*
So init obj val is -37000 so 37000 appears in TLC of init tab.

$$R = \begin{array}{|r|rrrrrr|} \hline 37000 & -80 & -80 & -80 & 0 & 0 & 0 \\ \hline 1300 & 16 & 0 & 0 & 1 & 0 & 0 \\ 2500 & 16 & 16 & 0 & 0 & 1 & 0 \\ 3700 & 16 & 16 & 16 & 0 & 0 & 1 \\ \hline \end{array}$$

Opt tab after 3 iters of Simplex:

$$R3 = \begin{array}{|r|rrrrrr|} \hline 55500 & 0 & 0 & 0 & 0 & 0 & 5 \\ \hline 325/4 & 1 & 0 & 0 & 3/50 & 0 & 0 \\ 75 & 0 & 1 & 0 & -3/50 & 3/50 & 0 \\ 75 & 0 & 0 & 1 & 0 & -3/50 & 3/50 \\ \hline \end{array}$$

Opt val of obj is -55000 so (remembering the orig prob was a max prob) opt revenue is 55500. $h_1 = 81.25, h_2 = h_3 = 75$. *Check that no eggs are produced* ($NE_1 = NE_2 = NE_3 = 0$).

Example 1.7 (Nurse Scheduling problem) *A hospital administrator is in charge of scheduling nurses to work each of the six shifts that start every four hours.*

- *The first shift starts at 08:00 (8 a.m.).*
- *Each shift lasts eight hours.*
- *The number of nurses required in each 4-hour time period is given in the Table on the next Slide.*
- *The problem is to schedule the nurses to meet the requirements and to do this while minimising the total number of nurses rostered to the six shifts.*
- *Once formulated as an LP, this problem is easily solved with the "Dual Simplex Method" — which I'll discuss later.*

Period of the Day	*Number of Nurses Required*
08:00–12:00	*140*
12:00–16:00	*120*
16:00–20:00	*160*
20:00–24:00	*90*
24:00–0400	*30*
04:00–08:00	*60*

(A solution will appear here.)

Solution: *DV's:* $x_1, \ldots, x_6$ *are the number of nurses allocated to each shift, starting @ 08:00.*

The IV's are the number available for each of the 6 time periods $n_1 = x_6 + x_1, n_2 = x_1 + x_2, \ldots, n_6 = x_5 + x_6$.

The constraints are just $n_i = R_i$ *where* R_i *are the required numbers in the RH column of the Table.*

Finally, $z = x_1 + \cdots + x_6$.

1.3 Exercises for Chapter 1

1. (a) In the T&C Corp problem, how low must the price of chairs fall before the optimal point $(2.25, 8)$ changes? What would the new optimal point be?

   (b) Suppose that the wholesale price of chairs spikes up to €2000 per chair! Will the optimal point change? Is there a value which, if the wholesale price of chairs exceeds it, causes the optimal point to change? Explain.

   (c) Are there positive values for the wholesale prices of tables & chairs so that only tables and not chairs will be produced in an optimal production plan?

2. Consider the LP

$$\max x_1 + x_2$$
$$\text{subject to}$$
$$-2x_1 + 2x_2 \leq 1$$
$$16x_1 - 14x_2 \leq 7$$
$$x_1, x_2 \geq 0.$$

(a) Use the graphical method to find an optimal point.

(b) Assume that in addition each variable is restricted to be integer valued. Can you find a feasible integer point by rounding the optimal x_1 & x_2 up or down?

(c) Use your graphical solution to find an integer optimal point.

3. (a) Can you solve this LP:

$$\begin{aligned} \max x_1 + 2x_2 + x_3 \\ \text{subject to} \\ x_1 + x_2 + x_3 \leq 4 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

simply by common-sense reasoning (no picture, no algebra)?

(b) If I replace the objective function by $x_1 + x_3$ can you describe the set of all possible solutions?

4. Consider the LP

$$\begin{aligned} \min -x_1 - x_2 \\ \text{subject to} \\ -x_1 + x_2 \leq 3 \\ x_1 - 2x_2 \leq 2 \\ x_1, x_2 \geq 0. \end{aligned}$$

(a) Use the graphical method to show that this LP is feasible but has no optimal point (unbounded).

(b) What if I add the constraint $2x_1 - 8x_2 \geq 4$?

5. Sales of tables & chairs have dropped — T&C Corp has diversified to produce wooden ornaments. T&C Corp uses three machines to produce two products. The Table shows the hours required on each machine to produce one unit of each product and the total time available on each machine during the production period.

	Time Used To Make		Total Machine Time Available in
Machine	Product 1	Product 2	Production Period
Lathe	1.1	2.0	1000
Sander	3.0	4.5	2000
Polisher	2.5	1.3	1500

- The Company wants to maximise the total number of (both) products made.
- Based on market research the Company needs to ensure that the amount produced of Product 1 is at least one third of the total number of (both) products made.

(a) Formulate a LP model for the problem.

(b) Suppose that instead of 3 machines & 2 products, I have 300 machines to make 200 products. (Wood is the future.)
  - Let r_{ij} stand for the number of hours on machine j required to produce one unit of product i.
  - Let c_j stand for the available machine time on machine j.

  Formulate a LP model for the enlarged problem.

6. A private coal-powered electrical generation plant burns three types of coal to produce electricity.
   - EU environmental regulations require that
     - ⋆ emissions from the smoke stack contain no more than 2500 parts per million (ppm) of Sulphur Dioxide SO_2
     - ⋆ and no more 15 kg/hour of particulate matter (smoke) be emitted from the smoke stack.
   - The Table gives the amounts of both pollutants that result from burning the three types of coal.

Coal Type	Sulphur Dioxide in Stack Emissions (ppm)	Particulates Emitted per Tonne of Coal Burned (kg/hr)
A	1200	1
B	2500	3
C	3700	2

- Burning coal A generates 22 MWh (Mega Watt hours) of electricity per hour.
- Burning coal B generates 26 MWh (Mega Watt hours) of electricity per hour.
- Burning coal C generates 32 MWh (Mega Watt hours) of electricity per hour.
- The plant has the capacity to burn 25 tonnes/hour of any mixture of the three coals.

- The total SO_2 ppm emissions from burning a mixture of coals are just the weighted average of the ppm emmisions of the individual coals — where each weight is the **proportion** of that coal used in the mixture.
- Similarly for the particulates emitted in kg/hr (here the total is used, not the weighted average).

Formulate a LP for operating the electric plant so as to maximise the amount of energy generated per hour.

7. (a) If you haven't already solved the Oil Refinery problem, Exercise 1.4 on Slide 43, do so now using the graphical method.

   (b) Are any of the constraints **redundant** — i.e. can any constraints be removed without changing the feasible set?

2 The Simplex Method

In Ch. 1, I was concerned with formulating LPs; translating from a verbal description to an algebraic description. In this Chapter I'll explain how the "Simplex Algorithm" can be used to solve any LP — if at least one feasible point exists.

2.1 Standard Form & Pivoting

The problems examined in Ch. 1 are all LP problems but came in slightly different forms.

- In this Section I will introduce a **Standard Form** into which any LP problem can be reformulated/translated.
- I will also describe the standard tabular representation for an LP — the **Simplex Tableau**.
- Finally, I will show how the Simplex Tableau for a Standard Form LP can be transformed using a method similar to Gaussian Elimination into a tableau which represents the solution.
- The transformed tableau has the same (optimal point) solution as the original — but the transformed tableau allows the solution to be “read off”.

2.1.1 Standard Form

A LP is said to be in **Standard Form** if it is:

- a minimisation problem
- with all equality constraints
- and all variables are required to be non-negative.

In algebraic (matrix) notation: (the scalar d is often omitted)

Definition 2.1 (Linear Program in Standard Form) *Given* $c \in \mathbb{R}^n$, *an* $m \times n$ *real matrix* A, $b \in \mathbb{R}^m$ *and a scalar (real number)* d *a Linear Program in Standard Form takes the form:*

$$\begin{aligned} \min_{x \in \mathbb{R}^n} c^T x + d \\ Ax = b \\ x \geq 0 \end{aligned} \tag{2.1}$$

Notation

- Throughout these Notes, I'll use capital letters $\mathbf{A}$ for matrices & lower case $\mathbf{c}$ for vectors.
- I will often (but not always) use a **bold font**, e.g. $\mathbf{c}$ to indicate a **vector**.
- The word "vector" will **always** mean a column of real numbers.
- A "row vector", written $\mathbf{c}^\mathsf{T}$ is the **transpose** of a (column) vector.
- The "dot product" of two (column) vectors $\mathbf{a}$ and $\mathbf{b}$, is often written $\mathbf{a} \cdot \mathbf{b}$.
- In these Notes, I'll use the "matrix" notation $\mathbf{a}^\mathsf{T}\mathbf{b}$ for $\mathbf{a} \cdot \mathbf{b}$.
- Here $\mathbf{a}^\mathsf{T}$ is a matrix consisting of a single row and $\mathbf{b}$ is a matrix consisting of a single column so $\mathbf{a}^\mathsf{T}\mathbf{b}$ is a 1×1 matrix!

Notation …

- So I can "unpack" $\mathbf{c}^T\mathbf{x}$ as $c_1x_1 + c_2x_2 + \cdots + c_nx_n$.
- Def. 2.1 can be "expanded" into **index notation** as:

Definition 2.2 (LP in Standard Form — Index Notation) *(Note that there are m equations in the n non-negative variables $x_1, x_2, \ldots, x_n$ and again the scalar d is often omitted.)*

$$\begin{array}{rrrrl}
\min\limits_{x_1,x_2,\ldots,x_n \in \mathbb{R}} & c_1x_1+ & c_2x_2+ & \ldots+ & c_nx_n + d \\
& \textit{subject to} & & & \\
& a_{11}x_1+ & a_{12}x_2+ & \ldots+ & a_{1n}x_n = b_1 \\
& a_{21}x_1+ & a_{22}x_2+ & \ldots+ & a_{2n}x_n = b_2 \\
& \vdots+ & \vdots+ & \vdots+ & \vdots = \vdots \\
& a_{m1}x_1+ & a_{m2}x_2+ & \ldots+ & a_{mn}x_n = b_m \\
& x_1, x_2, \ldots, x_n \geq 0. & & &
\end{array}$$

Notation ...

- The matrix notation used in Def. 2.1 is far more compact than the index notation used in Def. 2.2 above — you need to be comfortable with both.
- The i^{th} constraint equation is $a_i^T x = b_i$ where a_i^T is the i^{th} **row** of the matrix A.
- Remember that I am using the convention/rule that all vectors are columns, so to get a row vector (the i^{th} row of the matrix A) I need to take the transpose.
- In this notation $\mathbf{a}_i$ is the i^{th} row of the matrix A written as a column..

STOP **Stopped here** 10:00, Thursday Week 2

Notation ...

- I need an example!

  Example 2.1 *Suppose that I have the system of equations;*

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 6 \\ x_1 \; + \qquad + 5x_3 &= -2. \end{aligned}$$

This can be written

$$Ax = b, \textit{ where } A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \end{bmatrix}, \; b = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \textit{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

So, for example $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ *and* $\mathbf{a}_1^T \mathbf{x} = 2x_1 + 3x_2 - x_3$.

Notation . . .

- OK? Good..
- But just to check, can you see that $\mathbf{a}_2^\mathsf{T}\mathbf{x} = x_1 + 5x_3$?
- So $Ax = \begin{bmatrix} \mathbf{a}_1^\mathsf{T}\mathbf{x} \\ \mathbf{a}_2^\mathsf{T}\mathbf{x} \end{bmatrix}$, a **vector**.
- Remembering that two vectors are equal if and only if their corresponding entries are equal; the matrix equation $\mathbf{Ax} = \mathbf{b}$ translates into $\mathbf{a}_1^\mathsf{T}\mathbf{x} = b_1$ and $\mathbf{a}_2^\mathsf{T}\mathbf{x} = b_2$.
- Which of course is the same as

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 6 \\ x_1 \;+ \qquad\; + 5x_3 &= -2. \end{aligned}$$

Back to Standard Form

- To show how an LP can be manhandled into Standard Form , let's look again at the Dear Beer Co. problem; Example 1.3 from Ch.1.

- The problem, written as an LP is:

$$\max 6x_1 + 5x_2 + 3x_3 + 7x_4$$
$$\text{subject to}$$
$$1x_1 + 1x_2 + 0x_3 + 3x_4 \leq 50$$
$$2x_1 + 1x_2 + 2x_3 + 1x_4 \leq 150$$
$$1x_1 + 1x_2 + 1x_3 + 4x_4 \leq 80$$
$$x_1, x_2, x_3, x_4 \geq 0$$

- This LP is **not** in Standard Form as it is not a minimisation problem and the constraints are inequalities, not equalities.

- I can reformulate this LP as a minimisation problem by simply noting that:

  maximising $\mathbf{c}^\mathsf{T}\mathbf{x}$ subject to x feasible is equivalent to minimising $-\mathbf{c}^\mathsf{T}\mathbf{x}$ subject to x feasible.

- "**equivalent**" here means that the solution will be the same though of course the sign of the optimal $z = \mathbf{c}^\mathsf{T}\mathbf{x}$ is "flipped".

- Next, how do I turn the inequality constraints into equality constraints?

- Use the simple rule:

  $x \le y$ **is equivalent to** $x + s = y$ **where** $s \ge 0$.

- The new variable s is called a **slack variable**.

- Just apply this rule to each inequality constraint in turn:
  - $\star$ $1x_1 + 1x_2 + 0x_3 + 3x_4 \le 50 \rightarrow$
    $1x_1 + 1x_2 + 0x_3 + 3x_4 + \mathbf{1x_5} = 50.$
  - $\star$ $2x_1 + 1x_2 + 2x_3 + 1x_4 \le 150 \rightarrow$
    $2x_1 + 1x_2 + 2x_3 + 1x_4 + \mathbf{1x_6} = 150.$
  - $\star$ $1x_1 + 1x_2 + 1x_3 + 4x_4 \le 80 \rightarrow$
    $1x_1 + 1x_2 + 1x_3 + 4x_4 + \mathbf{1x_7} = 80.$
- There is a natural interpretation of the **slack variables** x_5, x_6 and x_7:

$$x_5 = \text{amount of hops not used}$$

$$x_6 = \text{amount of malt not used}$$

$$x_7 = \text{amount of yeast not used.}$$

- So the Dear Beer Co. problem can be written in Standard Form as:

$$\begin{aligned}
\min -6x_1 - 5x_2 - 3x_3 - 7x_4 + \mathbf{0x_5} + \mathbf{0x_6} + \mathbf{0x_7}& \\
\text{subject to}& \\
1x_1 + 1x_2 + 0x_3 + 3x_4 + \mathbf{1x_5} + \mathbf{0x_6} + \mathbf{0x_7} &= 50 \\
2x_1 + 1x_2 + 2x_3 + 1x_4 + \mathbf{0x_5} + \mathbf{1x_6} + \mathbf{0x_7} &= 150 \\
1x_1 + 1x_2 + 1x_3 + 4x_4 + \mathbf{0x_5} + \mathbf{0x_6} + \mathbf{1x_7} &= 80 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0
\end{aligned}$$

(I've written the **slack variables** x_5, x_6 & x_7 in **green**.)

- Notice that the new (slack) variables x_5, x_6 & x_7 have zero objective coefficients as I make no money from “hoarding” hops, yeast or malt.
- Slack variables will always have zero objective coefficients at the start of a problem in this course.
- Can you explain what it would mean if the objective coefficients for the slack variables were non-zero?

- The Standard Form of the Dear Beer Co. problem can be written in matrix/vector notation as

$\min_{x \in \mathbb{R}^7} c^T x + d$, subject to $Ax = b$ and $x \geq 0$ with

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 50 \\ 150 \\ 80 \end{bmatrix}, \; c = \begin{bmatrix} -6 \\ -5 \\ -3 \\ -7 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $d = 0$.

- In Section 2.9 I'll demonstrate the general procedure for reformulating any LP into standard form.
- The Dear Beer Co. LP has $\mathbf{d} = \mathbb{0}$ in the objective function $z = \mathbf{d} + \mathbf{c}^{\mathsf{T}}\mathbf{x}$ but I will show later that non-zero constant terms arise naturally when an LP is being solved.

2.1.2 The Simplex Tableau

Simplex Tableaux

- The Simplex tableau (plural tableaux) is a tabular representation of an LP in standard form.
- I will show that it also allows the current feasible point to be read off and a better feasible point found (the next iteration performed).

If an LP is in Standard Form (2.1) where the current objective value z is given by $z - \mathrm{d} = \mathbf{c}^T x$ (equivalent to $z = \mathrm{d} + \mathbf{c}^T \mathbf{x}$) , I can write its tableau (using vector notation) as:

$-d$	$\mathbf{c}^T(1 \times n)$
$\mathbf{b}(m \times 1)$	$\mathbf{A}(m \times n)$

More explicitly:

$-d$	c_1	c_2	$\dots$	c_n
b_1	A_{11}	A_{12}	$\dots$	A_{1n}
b_2	A_{21}	A_{22}	$\dots$	A_{2n}
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
b_m	A_{m1}	A_{m2}	$\dots$	A_{mn}

- The first row of the tableau represents the equation $z - d = c^T x$ (equivalent to $z = d + \mathbf{c}^T \mathbf{x}$) .
- The last m rows are the (equality) constraint rows.
- They represent the constraint equations $b_i = \mathbf{a}_i^T \mathbf{x}$, for $i = 1, \ldots, m$.
- The simplex tableau T_1 for the Dear Beer Co. problem is:

$T_1 =$

	Col. 0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Row 0	0	-6	-5	-3	-7	0	0	0
Row 1	50	1	1	0	3	1	0	0
Row 2	150	2	1	2	1	0	1	0
Row 3	80	1	1	1	4	0	0	1

- I will usually refer to the top row as **Row zero** and the LH column as column zero or the **constant column**.
- I usually won't bother to include the row & column labels (in **blue** in the Example).
- The same LP can be represented by many different but **equivalent** simplex tableaux.
- For example I might want to eliminate the variable x_1 from the second constraint equation (**Row 2** in T_1).
- (There are good reasons for doing this as I will explain shortly.)

- To do this I could solve the first equation for x_1 in terms of the other variables, getting $x_1 = 50 - x_2 - 3x_4 - x_5$.
- Then I could substitute this value for x_1 into the second constraint equation (**Row 2**).
- The second constraint equation becomes $150 = 2(50 - x_2 - 3x_4 - x_5) + x_2 + 2x_3 + x_4 + x_6$.
- Or just $50 = -x_2 + 2x_3 - 5x_4 - 2x_5 + x_6$.
- That was a bit tedious!
- An easier way to get this equation is to just add -2 times the first constraint equation to the second constraint equation.
- Or **equivalently** in tableau T_1, add -2 times **Row 1** to **Row 2**.
- **This is of course exactly what I do when using Gauss Elimination to solve a linear system.**

- I can eliminate x_1 from the third equation by adding -1 times the first equation to the third.
- Or **equivalently** in tableau T_1, add -1 times **Row 1** to **Row 3**.
- If I perform these two "row operations" on T_1, I get:

$T_2 =$

	Col. 0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Row 0	0	-6	-5	-3	-7	0	0	0
Row 1	50	1	1	0	3	1	0	0
Row 2	50	0	-1	2	-5	-2	1	0
Row 3	30	0	0	1	1	-1	0	1

- Now suppose that I want to eliminate the variable x_1 from the objective function (**again I will explain why shortly.**)

- I can do this by substituting $x_1 = 50 - x_2 - 3x_4 - x_5$ into the objective function and rewrite $z(\mathbf{x})$ as:
$z = -6(50 - x_2 - 3x_4 - x_5) - 5x_2 - 3x_3 - 7x_4.$

- Or just $z = -300 + x_2 - 3x_3 + 11x_4 + 6x_5.$

- Or **equivalently** in tableau T_1, add 6 times **Row 1** to **Row 0**.

- If I perform this third "row operation" on T_1, I get:

$T_2 =$

	Col. 0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Row 0	300	0	1	-3	11	6	0	0
Row 1	50	1	1	0	3	1	0	0
Row 2	50	0	-1	2	-5	-2	1	0
Row 3	30	0	0	1	1	-1	0	1

- **So what?**
- The tableau T_2 represents the same LP problem as does the tableau T_1.
- For example, the vector $x = (50, 0, 0, 0, 0, 50, 30)^T$ satisfies all three constraints represented by tableau T_2 and has $z = -300$.
- You can easily check that the same vector also satisfies all the constraints represented by tableau T_1 and also has $z = -300$.
- The "row operation" of adding a multiple of one row in a Tableau/Matrix to the others to eliminate a variable (introduce a zero in that variable's column) is called **pivoting**.
- The same term is used in Gauss Elimination as in Linear Programming.

2.1.3 Pivoting on a Simplex Tableau

- The sequence of row operations (just as in Gauss Elimination) performed on the simplex tableau T_1 to get tableau T_2 is called a **pivot**.

- **The steps in performing a pivot on a tableau are**:
  - ⋆ Select a non-zero entry in row r and column c of the matrix A.
  - ⋆ Row r is called the **pivot row** , column c is called the **pivot column** and the (r, c) position is the **pivot position**.
  - ⋆ Multiply the pivot row across by the constant $1/a_{rc}$ so that the entry in the pivot position equals 1.
  - ⋆ Use row operations to make all the other tableau entries in the **pivot column** equal to zero.

- The Simplex Algorithm for solving LP problems, introduced by George Dantzig in 1947, is a method for systematically choosing a sequence of pivots on a Standard Form Simplex Tableau.
- The pivots continue until a tableau in one of **four** final forms is found.
- In the Sections below I will show how, from the form of the final tableau, I can either write down an optimal point or conclude that no optimal point exists.

2.2 Pivoting Without Pain

- Solving any problem with more than 2–3 variables and 2–3 constraints just takes too much arithmetic for a hand solution to be practical.
- The good news is that I have provided you with a Matlab/Octave m-file `Pivot.m` that performs a single pivot.
- You need to decide which row & column to pivot on!
- See `http://jkcray.maths.ul.ie/ms4303/Pivot.m`.
- I used it to "do" all the examples in these Notes.
- I will demonstrate it in a tutorial — it will allow you to solve much larger problems than you could by hand.
- **But — you still need to be able to do pivots by hand!**

2.3 Canonical Form

I need to "transform" a Standard Form tableau into **canonical form** before I can solve the corresponding LP.

I'll explain with an Example. Suppose I have the following standard form simplex tableau.

Example 2.2 (Canonical Form Example)

	Col. 0	x_1	x_2	x_3	x_4	x_5
Row 0	-8	0	-2	7	0	0
Row 1	4	0	2	-4	1	0
Row 2	6	1	1	5	0	0
Row 3	5	0	-2	1	0	1

(It doesn't matter where it "came from", it could be the result of applying a couple of pivots to the tableau for a given LP problem.)

The important point is that from the form of the tableau, I can easily write down a feasible point!

- Just set $x_2 = 0$ and $x_3 = 0$ (**I will explain why shortly**).
- Then from the three constraint equations it follows that $x_1 = 6$, $x_4 = 4$ and $x_5 = 5$.
- This gives me the feasible vector $x = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 4 \\ 5 \end{bmatrix}$ with

  $z(x) = 8 - 2x_2 + 7x_3 = 8.$

2.3.1 Canonical Form Details

A Standard Form tableau is of limited use as it doesn't allow a point to be "read off". However, if a problem/tableau has the special property that the m columns of the $m \times m$ identity matrix appear as m of the n columns of A then I **can** "read off" the current point.

I need a new idea, a **Canonical Form** Simplex Tableau.

Definition 2.3 (Canonical Form) *A Simplex Tableau is in C.F. if:*

1. *the vector b has no negative elements*
2. *the m columns of the $m \times m$ identity matrix appear as m of the n columns of A (not necessarily in the right order)*
3. *the components of $\mathbf{c}$ corresponding to the m identity matrix columns are all zero.*

- You can easily check that the tableau in Example 2.2 above is in Canonical Form as:
  - ⋆ All the elements of $\mathbf{b}$ are non-negative.
  - ⋆ Cols 1, 4 and 5 of A form a shuffled version of the 3×3 identity matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
  - ⋆ N.B. The columns do not have to be in the "right" order.
  - ⋆ The objective function coefficients corresponding to cols 1, 4 and 5 of A, c_1, c_4 and c_5 are all zero.

- The variables corresponding to the m identity columns are called **basic variables** and the remaining variables are called **nonbasic variables**.
- **If a tableau is in canonical form then a feasible point can always be obtained by setting the nonbasic variables to zero and using the constraint equations to solve for the basic equations.**
- This works as there will only ever be one non-zero coefficient for a basic variable in each constraint row of a canonical form tableau.
- The feasible point found using this process is called a **basic feasible point**.

- So I can simply read off $x_{c_1} = b_{r_1}$, $x_{c_2} = b_{r_2}$, $x_{c_3} = b_{r_3}$, etc.where:
  - ⋆ c_1 is the first basic column and r_1 is the row of that column in which 1 appears,
  - ⋆ c_2 is the second basic column and r_2 is the row of that column in which 1 appears, etc.
  - ⋆ c_3 is the third basic column and r_3 is the row of that column in which 1 appears, etc.
- In the current example, Example 2.2 I can read off:
  - ⋆ $c_1 = 1$ and $r_1 = 2$ so $x_1 = b_2 = 6$.
  - ⋆ $c_2 = 4$ and $r_2 = 1$ so $x_4 = b_1 = 4$.
  - ⋆ $c_3 = 5$ and $r_3 = 3$ so $x_5 = b_3 = 5$.
- With a little practice this is very easy!

- In Example 2.2, the indices of the basic variables are 1, 4 and 5.
- Sometimes it is useful to list the indices of the basic variables in the order of the columns in which the corresponding columns of the identity matrix appear.
- For Example 2.2, the ordered list is $\mathcal{B} = (4, 1, 5)$.
- So $x_4 = b_1 = 4$, $x_1 = b_2 = 6$ and $x_5 = b_3 = 5$.
- A list $\mathcal{B}$ of basic indices ordered in this way is called a **basis**.

- Another example of a canonical form tableau:

Example 2.3 (A tableau in canonical form)

-9	0	-3	-7	0	0	4	0
2	0	2	1	1	0	-1	0
6	-1	-1	3	0	0	3	1
4	0	4	2	0	1	-2	0

- The **basis** is $\mathcal{B} = \{4, 7, 5\}$.
- So $x_4 = b_1 = 2$, $x_7 = b_2 = 6$ and $x_5 = b_3 = 4$.
- All the non-basic variables are set to zero.
- Finally, I can write the basic feasible point $x^T = (0, 0, 0, 2, 4, 0, 6)^T$.

STOP **Stopped here** 15:00, Monday Week 3

- And another tableau in canonical form:

Example 2.4 (Another tableau in canonical form)

0	0	0	−5	0	−7	−6	−3
50	0	0	1	1	3	1	0
180	0	1	1	0	1	2	2
80	1	0	1	0	4	1	1

- The **basis** is $\mathcal{B} = \{4, 2, 1\}$.
- $x_4 = b_1 = 50$, $x_2 = b_2 = 180$ and $x_1 = b_3 = 80$.
- All the non-basic variables are set to zero.
- So that the basic feasible point is $x^T = (80, 180, 0, 50, 0, 0, 0)^T$.

- So a canonical form tableau can easily be “decoded”, i.e. a basic feasible point can be read off.
- Some standard form tableaux cannot be transformed into canonical form.
- I will show later that these are exactly those tableaux corresponding to LPs that are infeasible (have no solution).
- I will show that there is always a series of pivots that can be applied to a standard form tableau that will transform it either into canonical form or an infeasible form (meaning that there is no solution).

A Point of Terminology

- Strictly speaking I should say that a **LP** is (or is not) in standard form but that a **tableau** is (or is not) in canonical form.
- In practice I will sometimes refer to a canonical form problem (meaning an LP whose Simplex Tableau is in canonical form,).
- But it doesn't make sense to refer to a standard form tableau (meaning the Simplex Tableau for a standard form LP) as a Simplex Tableau **by definition** only makes sense for a standard form LP.
- A better term is **starting** tableau.

2.3.2 Finding a Better Basic Feasible Point

- I'll display Example 2.2 again as I'll use it to illustrate the following.

	Col. 0	x_1	x_2	x_3	x_4	x_5
Row 0	-8	0	-2	7	0	0
Row 1	4	0	2	-4	1	0
Row 2	6	1	1	5	0	0
Row 3	5	0	-2	1	0	1

- The objective function is just $z(\mathbf{x}) = 8 - 2x_2 + 7x_3 = 8$, as the non-basic variables x_2 and x_3 are both zero.

- As I am interested in **reducing** $z(\mathbf{x})$ and because the coefficient of the nonbasic variable x_2 in the objective function is -2, I'll try increasing x_2 from zero as this will decrease $z(\mathbf{x})$ from 8.

- Let's set $x_2 = t > 0$ and leave $x_3 = 0$. (Increasing x_3 would increase $z(\mathbf{x})$ — the opposite of what I want.)
- If I substitute these values in the constraint equations in the above tableau then the basic variables shift in value to:
  - ⋆ $x_1 = 6 - t$,
  - ⋆ $x_4 = 4 - 2t$
  - ⋆ and $x_5 = 5 + 2t$.
- Now $z(\mathbf{x}(t)) = 8 - 2t$ so I want t as large as possible.
- But both x_1 and x_4 are **decreasing** as t increases so I cannot increase t indefinitely.
- Because all variables are required to be non-negative.
- To keep all variables non-negative, I must have:
  - ⋆ $6 - t \geq 0 \Leftrightarrow t \leq 6$,
  - ⋆ $4 - 2t \geq 0 \Leftrightarrow t \leq 2$

⋆ and $5 + 2t \geq 0 \Leftrightarrow t \geq -5/2$.

- So I can let t increase up to $t = 2$ and still keep the vector $\mathbf{x}(t)$ non-negative.
- Setting $t = 2$ gives the maximum possible decrease in $z(\mathbf{x}(t))$ while keeping $\mathbf{x}(t)$ non-negative.
- So I have the new vector (feasible point) $\hat{\mathbf{x}}$ where $\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 9 \end{bmatrix}$ with $z(\hat{\mathbf{x}}) = 4$ so z has been reduced.

- It would be nice if the new feasible point $\hat{\mathbf{x}}$ was a basic feasible point corresponding to a new canonical form tableau for the LP.
- For that to be possible,
  - ⋆ x_2 would have to be a basic variable in the new canonical form tableau
  - ⋆ and the corresponding Col 0 (constant column) entry would have to be 2.
- Let's re-examine the original tableau:

$T_1 =$

	Col. 0	x_1	x_2	x_3	x_4	x_5
Row 0	-8	0	-2	7	0	0
Row 1	4	0	2	-4	1	0
Row 2	6	1	1	5	0	0
Row 3	5	0	-2	1	0	1

- If I pivot on the circled entry in Row 1, Column 2, I get the tableau:

$$T_2 = \begin{array}{|c|r|rrrrr|}
\hline
 & \textbf{Col. 0} & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
\textbf{Row 0} & -4 & 0 & 0 & 3 & 1 & 0 \\
\hline
\textbf{Row 1} & 2 & 0 & 1 & -2 & 1/2 & 0 \\
\textbf{Row 2} & 4 & 1 & 0 & 7 & -1/2 & 0 \\
\textbf{Row 3} & 9 & 0 & 0 & -3 & 1 & 1 \\
\hline
\end{array}$$

- This tableau is in canonical form with respect to the basis $\mathcal{B} = (2, 1, 5)$.
- The associated basic feasible point is exactly the vector $\hat{\mathbf{x}}$.
- So it looks as if I can replace the parametric analysis with t by just doing a pivot on a carefully chosen element of the tableau.

- How do I choose the "right" element to pivot on?
- First note that for $t > 0$, $\mathbf{x}(t) = \begin{bmatrix} 6-t \\ 0+t \\ 0 \\ 4-2t \\ 5+2t \end{bmatrix}$ if and only if $t \leq 6/1$ and $t \leq 4/2$.
- So $\mathbf{x}(t) \geq 0$ if and only if $t \leq \min\{6/1, 4/2\}$.
- I can find this upper bound on t directly from the tableau as it is the minimum of the ratios $\dfrac{b_r}{a_{rk}}$ in column k for all the rows r having $a_{rk} > 0$.

- A row r in column k with $a_{rk} \leq 0$ cannot restrict t.
- For example $x_5(t) = 5 + 2t$ is nonnegative for all t.
- So the pivot row is the row corresponding to the upper limit on t, i.e. the row where the minimum ratio occurs.
- So I can write a general rule for the pivot row r selected.
- **The pivot row is the row r that minimises $\left\{ \frac{b_r}{a_{rk}} | a_{rk} > 0 \right\}$ for a given pivot column k.**
- For a given pivot column k there may be a tie for which row to select as pivot row (the same ratio may occur in several rows) — there are advanced methods for dealing with this situation.
- If I want, even if the selected pivot column has $c_k \geq 0$, I can pivot on the pivot row selected by the rule above. This can be useful as I will show later.

2.3.3 The Simplex Pivoting Rule

- If $c_k < 0$ it should be clear that
  - ⋆ if I pivot in a canonical form Simplex tableau
    - ∗ in column k
    - ∗ and in the pivot row selected by the rule on the previous Slide
  - ⋆ I will generate a new **canonical form** (**see Exercise 2.1 below for confirmation of this claim**) Simplex tableau
    - ∗ with a basic feasible point having an objective function less than or equal to the previous objective value.
  - ⋆ The objective function value will not change if the pivot row r has $b_r = 0$.

- I'll formally state this rule, the Simplex Pivot Rule.
  - ⋆ **Select the pivot column k so that $c_k < 0$.**
  - ⋆ **Select a row r s.t.:** $\dfrac{b_r}{a_{rk}} = \min_i \left\{ \dfrac{b_i}{a_{ik}} \quad \textbf{s.t. } a_{ik} > 0 \right\}$**.**
- In words;
  - ⋆ Select a column k with negative objective coefficient c_k.
  - ⋆ Select the row r with positive entry with the smallest "row ratio" $\dfrac{b_r}{a_{rk}}$.

Exercise 2.1 *Check carefully what happens to the b_i's as a result of a pivot using the Simplex Pivot Rule.* **Can I guarantee that they will not become negative (and make the Tableau non-canonical)**? *Hint: write down an algebraic formula for the value of b_i after a pivot has been performed on row r and column c using the Simplex Pivot Rule.*

An answer will appear here.

2.3.4 The Geometric Meaning of a Pivot

- An important idea: I will show with a simple example that pivoting on a canonical form tableau is equivalent to moving along the edge of the feasible region from one "corner" point to another.
- Consider yet again the T&C Corp example.

$$\max z = 20x_1 + 15x_2$$

$$\text{subject to}$$

$$\begin{aligned} x_2 &\le 8 \\ 2x_1 - x_2 &\le 0 \\ 2x_1 + x_2 &\le 12.5 \\ x_1, x_2 &\ge 0. \end{aligned}$$

- The feasible region is shaded in Figure 4, with the four corner points A,B,C & D.
- The feasible region is bounded by the lines $x_2 = 2x_1$, $2x_1 + x_2 = 12.5$, $x_2 = 8$, $x_1 = 0$ and $x_2 = 0$.

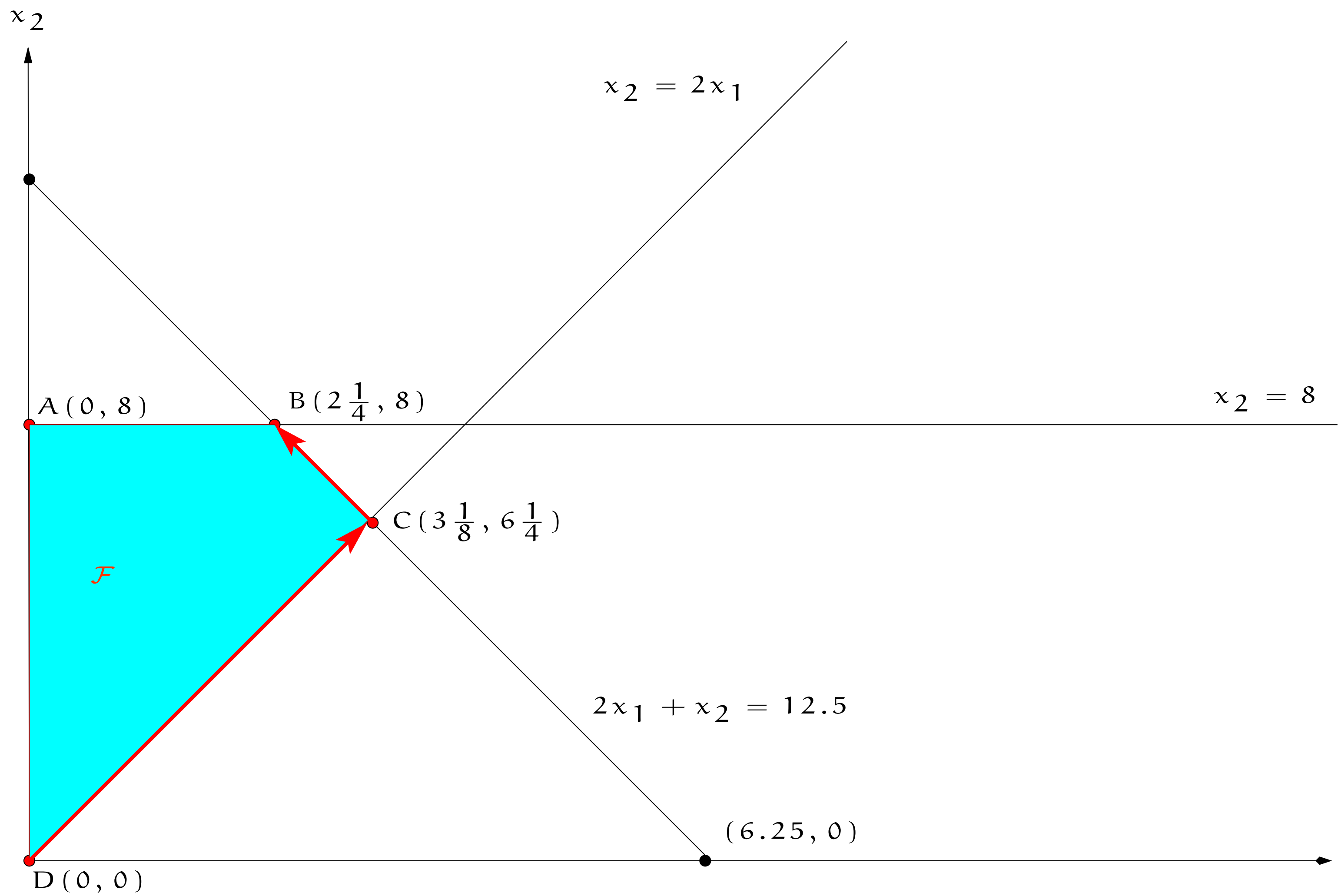

Figure 4: A feasible set with four corner pts.

- When I add slacks x_3, x_4, x_5 to turn the inequality constraints into equality constraints I have the tableau

$$T_1 = \begin{array}{|l|rrrrr|} \hline 0 & -20 & -15 & 0 & 0 & 0 \\ \hline 8 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 12.5 & 2 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

- Happily this is in canonical form — as I knew it would be.
- The basis is $\mathcal{B} = (3, 4, 5)$ and the basic feasible point is $\mathbf{x}^T = (0, 0, 8, 0, 0)$.
- The decision variables x_1 and x_2 are both nonbasic and therefore zero corresponding to the starting point D.

- If I apply the Simplex Pivot Rule from Slide 119, pivoting on the circled entry in T_1 I find (check)

$$\mathsf{T}_2 = \begin{array}{|l|ccccc|}\hline 0 & 0 & -25 & 0 & 10 & 0 \\ \hline 8 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -0.5 & 0 & 0.5 & 0 \\ 12.5 & 0 & 2 & 0 & -1 & 1 \\ \hline \end{array}$$

- The basis is $\mathcal{B} = (3, 1, 5)$ and the basic feasible point is $\mathbf{x}^\mathsf{T} = (0, 0, 8, 0, 12.5)$.
- The decision variables x_1 and x_2 are still both nonbasic and therefore zero corresponding to the starting point D.
- Geometrically: "I am still at the point D".
- And z has not changed — still zero.

- Again applying the Simplex Pivot Rule, pivoting on the circled entry in T_2, I find (check)

$T_3 =$

156.25	0	0	0	-2.5	12.5
1.75	0	0	1	**0.5**	-0.5
3.125	1	0	0	0.25	0.25
6.25	0	1	0	-0.5	0.5

- The basis is $\mathcal{B} = (3, 1, 2)$ and the basic feasible point is $\mathbf{x}^T = (3\frac{1}{8}, 6\frac{1}{4}, 1\frac{3}{4}, 0, 0)$.
- The decision variables x_1 and x_2 are $3\frac{1}{8}$ and $6\frac{1}{4}$ corresponding to the point C.
- The objective function z has decreased from 0 to $-156\frac{1}{4}$.

- Applying the Simplex Pivot Rule one more time, pivoting on the highlighted entry in T_3, I find (check)

$$T_4 = \begin{array}{|l|ccccc|} \hline 165 & 0 & 0 & 5 & 0 & 10 \\ \hline 3.5 & 0 & 0 & 2 & 1 & -1 \\ 2.25 & 1 & 0 & -0.5 & 0 & 0.5 \\ 8 & 0 & 1 & 1 & 0 & 0 \\ \hline \end{array}$$

- The basis is $\mathcal{B} = (4, 1, 2)$ and the basic feasible point is $\mathbf{x}^T = (2\frac{1}{4}, 8, 0, 3\frac{1}{2}, 0)$.
- The decision variables x_1 and x_2 are $2\frac{1}{4}$ and 8 corresponding to the optimal point B.
- The objective function z has decreased from $-156\frac{1}{4}$ to -165.
- The Simplex Pivot rule has selected the points D, D, C and B in succession.

- The objective function has either reduced or remained the same at each iteration.
- This is (very informally) the first example in these Notes of using the Simplex Pivot rule to solve an LP.
- There are details to be explained but the example is typical.

Stopped here 10:00, Thursday Week 3

2.4 Optimal, Unbounded & Infeasible Forms

- There are four possible final forms (patterns) for a **canonical form** simplex tableau.
- In this Section I'll describe these four forms and explain how to recognise & interpret them.

2.4.1 Optimal Form

- A vector x^* is a **minimising vector** if $z(\mathbf{x}) \geq z(x^*)$ for all feasible points $\mathbf{x}$.
- Here's Tableau T_2 from Slide 115 again:

$T_2 =$

	Col. 0	x_1	x_2	x_3	x_4	x_5
Row 0	-4	0	0	3	1	0
Row 1	2	0	1	-2	1/2	0
Row 2	4	1	0	7	-1/2	0
Row 3	9	0	0	-3	1	1

- I can "read off" a feasible vector/point $\hat{\mathbf{x}} = \begin{bmatrix} 4 & 2 & 0 & 0 & 9 \end{bmatrix}^{\mathsf{T}}$ with $z(\hat{\mathbf{x}}) = 4$, so the minimum value for the LP must be less than or equal to 4.
- To see if I can do better (reduce z) I only need to examine the tableau.
- I can read off the objective function as $z(x) = 4 + 3x_3 + 1x_4$.
- But any feasible x must be non-negative so $z(\mathbf{x}) \geq 4$ for any feasible x.
- So, as $z(\hat{\mathbf{x}}) = 4$, I can conclude that $\hat{\mathbf{x}}$ is a minimising vector for the LP.

- The reasoning depends only on the fact that the tableau is in canonical form and has cost vector (Row 0) coefficients $\mathbf{c} \geq 0$.
- In general if x^* is the basic feasible point for such a tableau, then $z(x^*) = \mathbf{d}$.
- Also, $z(\mathbf{x}) = \mathbf{d} + \mathbf{c}^\mathsf{T}\mathbf{x} \geq \mathbf{d}$ for any feasible $\mathbf{x}$ as $\mathbf{c} \geq 0$ and $\mathbf{x} \geq 0$ so $\mathbf{c}^\mathsf{T}\mathbf{x} \geq 0$.
- This means that $z(\mathbf{x}) \geq z(\hat{\mathbf{x}})$ for any feasible $\mathbf{x}$.
- In summary:

  Definition 2.4 (Optimal Form) *a tableau in canonical form is said to be in* **optimal form** *if* $\mathbf{c} \geq 0$.
- **The Definition only applies to tableaux in canonical form !**

2.4.2 Unbounded Form

- Not all tableaux in canonical form can be pivoted into optimal form.
- For example, consider the canonical form tableau:

	Col. 0	x_1	x_2	x_3	x_4	x_5
Row 0	-9	0	0	-2	-1	0
Row 1	3	0	0	-1	2	1
Row 2	1	1	0	0	1	0
Row 3	5	0	1	-4	1	0

- The basis $\mathcal{B}$ for this canonical form tableau is $\mathcal{B} = (5, 1, 2)$.
- The basic feasible point is $\mathbf{x} = \begin{bmatrix} 1 & 5 & 0 & 0 & 3 \end{bmatrix}^{\mathsf{T}}$.

- I'll increase the nonbasic variable x_3 to get a decrease in z.
- Let $x_3 = t > 0$ and $x_4 = 0$.
- Then $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 5+4t \\ 0+t \\ 0 \\ 3+t \end{bmatrix}$ with $z(\mathbf{x}(t)) = 9 - 2t$.
- By increasing t I can decrease z as much as I want.
- But here's the problem: $\mathbf{x}(t)$ stays non-negative for all $t \geq 0$.
- So the LP has no minimum value — it is **unbounded**.

- The reasoning depends only on the fact
  - ⋆ that the tableau is in canonical form ,
  - ⋆ has a negative cost vector coefficient in Row 0 ($c_3 = -2$ in this Example)
  - ⋆ and has no positive values in the selected column (the x_3 column in this Example).
- In summary:

  Definition 2.5 (Unbounded Form) *a tableau in canonical form is said to be in* **unbounded form** *if* $c_k < 0$ *for some column* k *and* $a_{ik} \leq 0$ *for each row* i *(no positive values in column* k*).*

- **The Definition only applies to tableaux in canonical form !**

2.4.3 Two Infeasible Forms

- If a tableau is in canonical form, then I know that the LP has at least one feasible point, the basic feasible point associated with that canonical form.
- But I'll show that some LPs in standard form cannot be transformed into canonical form.
- One reason is that not every "linear system" of equations has a solution.
- The first infeasible form for an LP follows from this fact.

- For example, look at this tableau for a standard form LP:

$$T_1 = \begin{array}{|r|rrrr|}
\hline
 & x_1 & x_2 & x_3 & x_4 \\
\hline
5 & 2 & -3 & 1 & -1 \\
\hline
2 & 1 & 0 & -1 & -1 \\
1 & -1 & 1 & 2 & 0 \\
-4 & 0 & 1 & 1 & -1 \\
\hline
\end{array}$$

- It is not in canonical form, so I could try a sequence of pivots to transform it to canonical form.
- (I'll explain this technique in a more structured way in Section 2.6.)
- (Remember that a pivot does not change the LP or its solution, it just shuffles the equations around.)

- Pivoting on the circled element of T_1 gives:

$$T_2 = \begin{array}{|c|cccc|}
\hline
 & x_1 & x_2 & x_3 & x_4 \\
\hline
1 & 0 & -3 & 3 & 1 \\
\hline
2 & 1 & 0 & -1 & -1 \\
3 & 0 & 1 & 1 & -1 \\
-4 & 0 & 1 & 1 & -1 \\
\hline
\end{array}$$

- Pivoting on the circled element of T_2 gives:

$$T_3 = \begin{array}{|r|rrrr|} \hline & x_1 & x_2 & x_3 & x_4 \\ \hline 10 & 0 & 0 & 6 & -2 \\ \hline 2 & 1 & 0 & -1 & -1 \\ 3 & 0 & 1 & 1 & -1 \\ -7 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

- But this is infeasible as the third constraint equation translates into: $-7 = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0$ which is not satisfied by **any** vector $\mathbf{x}$!

- Now generalise to any standard form tableau as follows: if the system of equations $Ax = b$ has solutions then no such row could be obtained by row operations/pivots.
- So if a standard form tableau can be transformed by pivoting into a tableau with a row of this form then the LP is infeasible.
- In summary:

  Definition 2.6 (First Infeasible Form) *A tableau in standard form is in* **first infeasible form** *if some row* r *has* $b_r \neq 0$ *and* $a_{rk} = 0$ *for each column* k *(*r^{th} *row of* A *is* **all zeros***).*

Even if a system of equations $Ax = b$ has solutions, they must be non-negative to be **feasible** (SF LP: $x_i \geq 0, i = 1, \ldots, n$).

- For example the LP in standard form with tableau:

	x_1	x_2	x_3	x_4
-6	-1	1	-1	0
8	2	-2	-6	0
-2	0	5	4	1

  has the second constraint equation: $-2 = 5x_2 + 4x_3 + x_4$.

- Obviously no non-negative vector $\mathbf{x}$ can satisfy this equation.

- In summary:

  Definition 2.7 (Second Infeasible Form) *A tableau in standard form is in* **second infeasible form** *if some row r has $b_r < 0$ and $a_{rk} \geq 0$ for each column k (r^{th} row of A is non-negative).*

An Outline Procedure For Solving an LP

- Represent your Standard Form LP as a Simplex tableau (see Section 2.9 below).
- Phase 1:
  - ⋆ Use pivots to transform the starting tableau into canonical form.
  - ⋆ If a pivot produces a tableau in either infeasible form, STOP — the LP is infeasible.
- Phase 2:
  - ⋆ Now apply pivots to the canonical form tableau.
  - ⋆ If a pivot produces a tableau in unbounded form, STOP — the LP is unbounded.
  - ⋆ If a pivot produces a tableau in optimal form, STOP — an optimal point (solution) is found.

Looking Ahead

- The Outline Procedure is still a little vague – I will fill in the details in the next Sections.
- First I'll explain how to solve LPs whose tableau is in canonical form (Phase 2).
- Then I'll show how to transform the standard form tableau into canonical form, (Phase 1).
- Finally, I'll show how to represent the LP as a standard form tableau (see Section 2.9 below).

2.5 Solving LPs in Canonical Form

2.5.1 Pivoting to Optimal Form

- When a LP has a tableau in canonical form there are only three possibilities; the tableau is either (**check**);
  - ⋆ in optimal form
  - ⋆ in unbounded form
  - ⋆ or there is some column k such that c_k is negative and at least one row r in column k such that $a_{rk} > 0$.
- In the last case one or more pivots with the Simplex Rule will lead to optimal or unbounded form.
- There are rare cases where "cycling" can occur (the method repeatedly steps though the same set of basic feasible points).
- There are methods for avoiding this problem — I won't give them in these Notes.

- Here's an example of a typical sequence of simplex rule pivots that leads to optimal form — starting with a canonical form tableau.
- I've circled the pivot positions in each tableau.
-

	x_1	x_2	x_3	x_4	x_5	x_6
-6	0	-1	-4	0	0	0
2	1	-3	1	0	0	1
20	1	4	-1	1	0	0
1	1	-1	1	0	1	0

•

	x_1	x_2	x_3	x_4	x_5	x_6
-2	4	-5	0	0	4	0
1	0	-2	0	0	-1	1
21	2	3	0	1	1	0
1	1	-1	1	0	1	0

	x_1	x_2	x_3	x_4	x_5	x_6
33	22/3	0	0	5/3	17/3	0
15	4/3	0	0	2/3	-1/3	1
7	2/3	1	0	1/3	1/3	0
8	5/3	0	1	1/3	4/3	0

- In the first tableau I could have chosen either the x_2 or the x_3 column as pivot column.

- In this course I will use the simple rule **pick the column with the most negative objective coefficient**.

- Other rules sometimes find an optimal tableau in fewer pivots.

Degenerate LPs

- A technical point: if a LP has one or more zero $\mathtt{b_r}$'s in one of its canonical form tableaux, it is called **degenerate**.
- This means that at least one basic variable is zero (can you see why?).
- A non-degenerate LP will always converge, provided it is feasible & bounded.
- For a non-degenerate LP, the simple column selection rule above will always work.
- For a degenerate LP, cycling may occur if the simple column selection rule is used — though this behaviour is very rare.
- Enough!

2.6 From Standard Form to Canonical Form

- I have explained how to pivot any canonical form tableau to either optimal or unbounded form.
- Now I need to examine the problem of getting from the initial standard form tableau for an LP to a canonical form tableau — if this is possible.
- If not, then to one of the two infeasible forms.
- The process is often referred to as **Phase 1**.
- I'll break it into two parts, Phase 0 & Phase 1.

2.6.1 Getting an Identity — Phase 0

- Remember that for a tableau to be in canonical form the matrix A must contain the columns of the $m \times m$ identity matrix and the cost coefficients associated with these columns must be zero.
- So I'll start the process of transforming a problem into canonical form by pivoting to get an identity matrix in A.
- Start with the following non-canonical form tableau:

	x_1	x_2	x_3	x_4	x_5
0	1	1	0	0	-2
1	1	0	-1	2	-2
4	-1	-1	2	-1	1
5	0	-1	1	1	-1

- Pivoting on the circled element, I have

	x_1	x_2	x_3	x_4	x_5
-1	0	1	1	-2	0
1	1	0	-1	2	-2
5	0	-1	1	1	-1
5	0	-1	1	1	-1

- Pivoting on the circled element -1, I have

	x_1	x_2	x_3	x_4	x_5
4	0	0	2	-1	-1
1	1	0	-1	2	-1
-5	0	1	-1	-1	1
0	**0**	**0**	**0**	**0**	**0**

- The row of all zeros in the last tableau shows that the last constraint is **redundant**.
- It is a combination of the other constraint equations.
- I can drop this last row from the tableau, leaving the tableau:

	x_1	x_2	x_3	x_4	x_5
4	0	0	2	-1	-1
1	1	0	-1	2	-1
-5	0	1	-1	-1	1

- This tableau still isn't in canonical form but the columns of the identity matrix are in place with zeros above them.
- The only thing that could have gone wrong is that I could have got a tableau with a non-zero b_r in the constant column and all zero a_{rk} in the remainder of row r.
- But that would mean the tableau was in infeasible form 1.

- I'll call this process of "getting an identity" **Phase 0** — summarised as:

Definition 2.8 (Phase 0) *The process:*

1. *Set row* $r = 1$
2. *For any row* $r \geq 1$
   - ⋆ *Find a non-zero* a_{rk} *in row* r.
   - ⋆ *IF row r of the tableau all zero (redundant row), delete the row and go to step 3.*
   - ⋆ *IF each* a_{rk} *is zero but* b_r *non-zero, STOP in infeasible form 1.*
   - ⋆ *Otherwise pivot on the first non-zero entry in row* r *of* A.
3. *IF* $r < n$ *set* $r = r + 1$ *and go to step 2, otherwise STOP.*

- This process either
  - ⋆ produces a tableau that has the m identity columns appearing as columns with the corresponding cost coefficients equal to zero
  - ⋆ or a tableau in infeasible form 1.

2.7 Using the Dual Simplex Method to Transform a Problem to Canonical Form

What's the Dual Simplex Method? I haven't discussed it yet — I will derive it in Section 3.5.

In this Section I will use it as a method to transform a tableau to which Phase 0 has been applied (so the columns of the identity matrix are in place with zeros above them) into canonical form.

I'll treat the Dual Simplex Method (DSM) as a tool to achieve this without worrying (for now) where the method comes from.

In fact we will see in Section 3.5 that (as you might guess) DSM is based on the Simplex Method — but the Simplex Method applied to a transformed version of the tableau to be worked on.

Definition 2.9 (Simple DSM) *The DSM algorithm is stated formally on Slide 260. Here is a simplified description.*

1. *If the tableau has one or more negative entries in Column 0 (the constant column)*
   - *Find the row with the most negative element in Column 0 — label it the* **pivot row**.
   - *Considering only the columns with* **negative** *entries in the pivot row (to the right of Column 0) calculate the* **column ratio** *— the ratio of the number "at the top of the column" to the number in the pivot row & the current column.*
   - *Find the column with the largest/most positive/least negative* **column ratio** *and call it the* **pivot column**.
   - *Pivot on the* **pivot entry**, *the entry in the intersection of the pivot row & pivot column.*
2. *Repeat Step 1 until all entries in Column 0 are non-negative.*

Let's see an example.

	x_1	x_2	x_3	x_4	x_5
0	-3	-2	0	0	0
-3	-3	-1	1	0	0
-6	-4	-3	0	1	0
3	1	1	0	0	1

This tableau is not in canonical form as, though the columns of the identity matrix are in place with zeros above them, the numbers in Column 0 are not all non-negative. In fact I can read off the vector x as $(0, 0, -3, -6, 3)$ which does not satisfy the non-negativity condition for a Standard Form LP.

The objective/cost coefficients are not all non-negative so the vector x is infeasible (as not non-negative) and sub-optimal (as cost coefficients not all not non-negative).

Let's apply the DSM:

- Find the row with the most negative element in Column 0. The third row of the tableau (Row 2) has -6 in Column 0 so it is the **pivot row**.
- Columns 1 & 2 (the x_1 and x_2 columns) have negative entries, -4 and -3.
- The column ratios for these two columns are $3/4$ and $2/3$ respectively. I select Column 1 for the **pivot column** as it has the larger column ratio.
- Now pivot on this entry of the tableau .

I find (check) that after the pivot I have

	x_1	x_2	x_3	x_4	x_5
$4\frac{1}{2}$	0	$\frac{1}{4}$	0	$-\frac{3}{4}$	0
$1\frac{1}{2}$	0	$1\frac{1}{4}$	1	$-\frac{3}{4}$	0
$1\frac{1}{2}$	1	$\frac{3}{4}$	0	$-\frac{1}{4}$	0
$1\frac{1}{2}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	1

which is in canonical form (but still not optimal).

One iteration of the Simplex Method transforms the tableau into optimal form and the solution can be read off as: $(3, 0, 6, 6, 0)$ with $z = -9$. (Check.)

STOP **Stopped here** 15:00, Monday Week 4

Let's try a more complicated example.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
81	0	0	0	2	14	-42	13	0	-12
-55	0	0	0	0	-5	25	-6	1	5
-4	1	0	0	-1	0	10	0	0	1
2	0	1	0	1	-1	1	-1	0	0
1	0	0	1	-1	2	-11	2	0	-1

Again, this tableau is not in canonical form as, though the columns of the identity matrix are in place with zeros above them (the x_8, x_1, x_2 and x_3 columns), the numbers in Column 0 are not all non-negative. I can read off the vector x as $(-4, 2, 1, 0, 0, 0, 0, -55, 0)$ which does not satisfy the non-negativity condition for a Standard Form LP.

Again let's apply the DSM:

- Find the row with the most negative element in Column 0. The second row of the tableau (Row 1) has -55 in Column 0 so it is the **pivot row**.
- Columns 5 & 7 (the x_5 and x_7 columns) have negative entries, -5 and -6.
- The column ratios for these two columns are $-14/5$ and $-13/6$ respectively. I select Column 7 for the **pivot column** as it has the larger/less negative column ratio.
- Now pivot on this entry of the tableau .

I find

$T_1 =$

	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$	$\mathbf{x_8}$	$\mathbf{x_9}$
$-38\frac{1}{6}$	0	0	0	2	$3\frac{1}{6}$	$12\frac{1}{6}$	0	$2\frac{1}{6}$	$-1\frac{1}{6}$
$9\frac{1}{6}$	0	0	0	0	$\frac{5}{6}$	$-4\frac{1}{6}$	1	$-\frac{1}{6}$	$-\frac{5}{6}$
-4	1	0	0	-1	0	10	0	0	1
$11\frac{1}{6}$	0	1	0	1	$-\frac{1}{6}$	$-3\frac{1}{6}$	0	$-\frac{1}{6}$	$-\frac{5}{6}$
$-17\frac{1}{3}$	0	0	1	-1	$\frac{1}{3}$	$-2\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$

Unfortunately I still have negative elements in Column 0 though they have decreased in magnitude.

Again let's apply the DSM:

- Find the row with the most negative element in Column 0. The last row of the tableau (Row 4) has $-17\frac{1}{3}$ in Column 0 so it is the **pivot row**.
- Columns 4 & 6 (the x_4 and x_6 columns) have negative entries, -1 and $-2\frac{2}{3}$.
- The column ratios for these two columns are -2 and $-73/16$ respectively. I select Column 4 for the **pivot column** as it has the larger/less negative column ratio.
- Now pivot on this entry of the tableau .

I find

$$T_2 = \begin{array}{|r|rrrrrrrrr|}
\hline
 & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{x_5} & \mathbf{x_6} & \mathbf{x_7} & \mathbf{x_8} & \mathbf{x_9} \\
\hline
-72\frac{5}{6} & 0 & 0 & 2 & 0 & 3\frac{5}{6} & 6\frac{5}{6} & 0 & 2\frac{5}{6} & \frac{1}{6} \\
\hline
9\frac{1}{6} & 0 & 0 & 0 & 0 & \frac{5}{6} & -4\frac{1}{6} & 1 & -\frac{1}{6} & -\frac{5}{6} \\
13\frac{1}{3} & 1 & 0 & -1 & 0 & -\frac{1}{3} & 12\frac{2}{3} & 0 & -\frac{1}{3} & \frac{1}{3} \\
-6\frac{1}{6} & 0 & 1 & 1 & 0 & \frac{1}{6} & -5\frac{5}{6} & 0 & \frac{1}{6} & -\frac{1}{6} \\
17\frac{1}{3} & 0 & 0 & -1 & 1 & -\frac{1}{3} & 2\frac{2}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\
\hline
\end{array}$$

Still not in canonical form but only one (smaller) negative entry left in Column 0.

One more iteration of DSM gives (check):

$T_3 =$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
−79	0	1	3	0	4	1	0	3	0
40	0	−5	−5	0	0	25	1	−1	0
1	1	2	1	0	0	1	0	0	0
37	0	−6	−6	0	−1	35	0	−1	1
42	0	−4	−5	1	−1	26	0	−1	0

This is in canonical form and is also in optimal form, an unexpected bonus!

2.7.1 Could DSM Undo the Phase 0 Procedure?

One possible difficulty;

- could the process of repeatedly applying DSM undo the "the columns of the $m \times m$ identity matrix are in place with zeros above them" property introduced by Phase 0?
  - ⋆ where m is the number of constraint rows.

If this could happen then the DSM iterations are useless as the tableau will not be in canonical form.

The answer is no because:

- the "1" in such a column cannot be selected as the pivot element for DSM as the pivot element must be negative

  and

- A DSM pivot will not affect a "column from the identity matrix" if the pivot row does not coincide with the row in which the "1" occurs.

- A DSM pivot will demote a "column from the identity matrix" to non-basic status if the pivot row coincides with the row in which the "1" occurs.

- It will introduce non-zero elements into the “column from the identity matrix” (and change the “1”to a negative value)
  - ⋆ So the **basic** “column from the identity matrix”becomes non-basic.
  - ⋆ But the “column from the identity matrix” reappears in the pivot column.
- The other “columns from the identity matrix” are unaffected — can you see why?

2.7.2 Does DSM Always Work?

DSM stops when there are no more negative elements in Column 0 (of course the z–value in the top left position is not considered).

Suppose that there is still at least one negative element left in Column 0, in Row R say — how could DSM fail?

- The method looks at the negative elements of Row R and calculates the column ratio as explained previously.
- But what if there are no negative elements in this row?

- There are two possibilities (remember that there is still at least one negative element left in Column 0, in Row R):
  - ⋆ Either all the elements of Row R to the right of Column 0 are zero — the tableau is in **first infeasible form** (see the definition of the first infeasible form Def. 2.6 on Slide 139) — **but this cannot happen by the discussion in Sec. 2.7.1 above** — **why?**.
  - ⋆ Otherwise there is at least one positive element in Row R to the right of Column 0 — the tableau is in **second infeasible form** (see the definition of the second infeasible form Def. 2.7 on Slide 140).
- So DSM fails if the problem is infeasible, i.e. there is no solution.
- **Check** that a similar argument shows that the Simplex Method (SM) only fails if the problem is unbounded.

2.8 The Simplex Algorithm

Now that I can transform a standard form LP into canonical form, I can assemble the pieces into an "algorithm" or procedure.

The Simplex Algorithm Given a LP in standard form with simplex tableau:

	$x_1 \; x_2 \; \ldots \; x_n$
$-d$	$\mathbf{c}^T$
$\mathbf{b}$	A

where A is $m \times n$.

- PHASE 0
  - ⋆ Pivot using the Phase 0 procedure (see Def. 2.8) to introduce the columns of the identity matrix with zeros in those columns in the top row.
  - ⋆ If the resulting tableau is in infeasible form 1 then STOP.
  - ⋆ Otherwise go to Phase 1.
- PHASE 1
  - ⋆ Pivot using the DSM (See Def. 2.9) to get an initial canonical form .
  - ⋆ If tableau is in infeasible form 2 then STOP.
  - ⋆ Otherwise the resulting tableau is in canonical form — go to Phase 2.

- PHASE 2

  Repeat the following steps until an optimal or unbounded form tableau is found.

  ⋆ If the current canonical form tableau has $c_j \geq 0$ for $j = 1, \ldots, n$ then the current basic feasible point is optimal. STOP.

  ⋆ Otherwise select a column k with $c_k < 0$.

  ⋆ If $a_{rk} \leq 0$ for all $r = 1, \ldots, m$ then **the LP is unbounded**. **STOP**.

  ⋆ Otherwise find the row r (with $a_{rk} > 0$) such that the ratio b_r/a_{rk} is smallest.

  ⋆ Pivot in row r and column k to update the tableau.

That's it — that's the Simplex Algorithm.

2.9 Reformulating Any LP into Standard Form

- A loose end — how do I transform an LP into standard form ?
- Fortunately this is always possible and easy.

1. **Max to min.**
   - To convert a max problem to a min problem just multiply **c** by -1.
   - For example the solution to

$$\max \mathbf{c}^T x \quad \text{s.t.} \;\; A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$$

   is the solution to

$$\min -\mathbf{c}^T x \quad \text{s.t.} \;\; A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0.$$

   - Of course the optimal z value of the second LP is -1 times the optimal z value of the first.

2. **Inequality Constraints to Equality Constraints**

- I showed previously that an inequality constraint can be replaced by an equality constraint if an extra non-negative "slack" variable is added to the problem.
- A extra column has to be added to the tableau for each slack variable.
- The cost coefficient (at the top of each slack column) is zero as the objective function does not depend on the slack variables.

- Here's an example.

$$
\begin{aligned}
\min 3x_1 - x_2 + 4x_3 \\
\text{subject to} \\
x_1 + x_2 + 2x_3 \le 10 \\
x_1 + 0x_2 - x_3 = 5 \\
2x_1 - x_2 - x_3 \ge 8 \\
x_1, x_2, x_3 \ge 0.
\end{aligned}
$$

transforms to

$$\min 3x_1 - x_2 + 4x_3 + 0x4 + 0x5$$

$$\text{subject to}$$

$$x_1 + x_2 + 2x_3 + x_4 = 10$$

$$x_1 + 0x_2 - x_3 = 5$$

$$2x_1 - x_2 - x_3 - x_5 = 8$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

- I also have the option of writing the last constraint as $-2x_1 + x_2 + x_3 + x_5 = -8$.
- This is often preferred as the extra columns are then columns from the identity matrix.

3. **Free Variables**

- Variables in LPs are not always be restricted to being non-negative.
- For example, a variable x might represent the balance in a bank account — which could be negative.
- Such variables are called **free variables**.
- **The remedy is simple, just replace each free variable x_k by the difference of two non-negative variables $x_k = x'_k - x''_k$.**

- For example :

$$\min 3x_1 - x_2 - 4x_3$$
$$\text{subject to}$$
$$x_1 + x_2 + 2x_3 = 10$$
$$x_1 - 5x_2 - 2x_3 = 7$$
$$x_1 \geq 0,\ x_2, x_3 \text{ free.}$$

transforms to

$$\min 3x_1 - (x_2' - x_2'') - 4(x_3' - x_3'')$$
$$\text{subject to}$$
$$x_1 + (x_2' - x_2'') + 2(x_3' - x_3'') = 10$$
$$x_1 - 5(x_2' - x_2'') - 2(x_3' - x_3'') = 7$$
$$x_1, x_2', x_2'', x_3', x_3'' \geq 0.$$

- In practice, I would label x_2' and x_2'' as x_2 & x_3.
- Similarly I would label x_3' and x_3'' as x_4 & x_5.
- Giving the LP:

$$\begin{aligned}
\min 3x_1 - x_2 + x_3 - 4x_4 + 4x_5 & \\
\text{subject to} & \\
x_1 + x_2 - x_3 + 2x_4 - 2x_5 &= 10 \\
x_1 - 5x_2 + 5x_3 - 2x_4 + 2x_5 &= 7 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0.
\end{aligned}$$

- If I have a feasible point for the original problem it is easy to construct one in terms of the new variables — can you see how?
- And finally, when I find an optimal solution to the transformed problem it is easy to revert to the original variables.

- This procedure for dealing with free variables has one drawback, I get an extra variable for every free variable.
- A clever way to avoid this is to notice that any vector can be written as the difference of two nonnegative vectors, with one of the vectors having all elements the same.
- For example:

$$\begin{bmatrix} -7 \\ 2 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 2 \\ 11 \end{bmatrix} - \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}$$

- The second vector just repeats the largest of the absolute values of the negative components of the original vector — w say.
- So $\mathbf{x} = \mathbf{y} - w\mathbf{e}$ where $\mathbf{e}$ is a constant vector of ones.
- With a little thought the original LP (with free variables) can be rewritten in terms of the non-negative vector $\mathbf{y}$ and the single non-negative variable w.
- **Exercise 2.2** *Explain how.*

Final Comment on the Simplex Algorithm

- Final comment: the tableau approach to solving LPs is not used in practice.
- Methods based on matrix algebra are used instead — more in Third Year (MS4105 Linear Algebra 2) and Fourth Year (MS4327 Optimization).

Stopped here 10:00, Thursday Week 4

2.10 Exercises for Chapter 2

1. Consider the following LP:

$$\max 2x_1 + 3x_2$$

$$\text{subject to}$$

$$x_1 + x_2 \leq 4$$

$$-x_1 + 2x_2 \geq -1$$

$$x_1, x_2 \geq 0.$$

   (a) Reformulate into a standard form LP.

   (b) Enter the data for the reformulated problem into a simplex tableau and pivot (if necessary) to get an initial canonical form tableau.

   (c) Perform simplex rule pivots on the tableau of part (b) to solve the LP. (Use `Pivot.m` to check your work if you like.)

2. Solve the following LP:

$$\min -2x_1 + x_2$$

$$\text{subject to}$$

$$x_1 + 2x_2 \le 10$$

$$x_1 - x_2 \le -5$$

$$x_1, x_2 \ge 0.$$

using the Simplex Algorithm. (Again, use `Pivot.m` to check your work if you like.

3. The following tableau represents a LP in standard form.

	x_1	x_2	x_3	x_4	x_5
-9	0	b	e	0	0
a	1	c	1	0	0
2	0	d	-1	1	0
4	0	-1	1	0	1

Give conditions on the parameters a, b, c, d and e so that

(a) the tableau is in optimal form

(b) the tableau is in unbounded form

(c) the tableau is in infeasible form 2

(d) the tableau is in optimal form and the feasible region (not the tableau) is unbounded.

4. (a) Reformulate the following LP into standard form:

$$\max 5x_1 + 7x_2 - 2x_3 + 3x_4 - 6x_5$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

(b) Perform a single pivot that will result in an initial canonical form that is also optimal.

(c) Can you come up with a rule for simply writing down an optimal solution to any LP of the form

$$\max \mathbf{c}^T\mathbf{x} \quad \text{s.t.}\, x_1 + x_2 + \cdots + x_n = 1 \quad \text{with } \mathbf{x} \geq 0?$$

(Easier than it looks.)

5. (a) Solve the following LP:

$$\min 2x_1 - 7x_2 - 3x_3$$

$$\text{subject to}$$

$$x_1 + 2x_2 + x_3 \le 5$$

$$2x_1 + 0x_2 + x_3 \le 10$$

$$x_1, x_2, x_3 \ge 0.$$

(b) Is this LP degenerate? (See Slide 147 for a Definition.)

6. Solve the following LP:

$$
\begin{aligned}
\max -2x_1 - x_2 + 4x_3 \\
\text{subject to} \\
3x_1 - x_2 + 2x_3 \le 25 \\
-x_1 - x_2 + 2x_3 \le 20 \\
-x_1 - x_2 + x_3 \le 5 \\
x_1, x_2, x_3 \ge 0.
\end{aligned}
$$

7. Consider the following canonical form LP:

$$\min 2x_1 - x_2 + 0x_3 + 3x_4 + 0x_5$$

$$\text{subject to}$$

$$x_1 + x_2 + 0x_3 + x_4 + x_5 = 10$$

$$3x_1 - x_2 + x_3 - 2x_4 + 0x_5 = 6$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

(a) Write a canonical form simplex tableau T that represents this LP.

(b) What is the basis $\mathcal{B}$ associated with this tableau?

(c) Write the basic feasible point associated with the tableau T?

8. Use the DSM to find an initial canonical form for the LP:

$$
\begin{aligned}
\min 2x_1 + x_2 + x_3 + x_4 \\
\text{subject to} \\
x_1 + 2x_2 + 0x_3 + x_4 = 5 \\
x_1 + x_2 + 2x_3 + x_4 = 10 \\
x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$

9. Now solve the problem to optimality.

10. The following tableau is the initial canonical form tableau for the Dear Beer Co..

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	-6	-5	-3	-7	0	0	0
50	1	1	0	3	1	0	0
150	2	1	2	1	0	1	0
80	1	1	1	4	0	0	1

Find an optimal form for this LP.

11. Solve the following standard form LPs.

- You will need `Pivot.m`!!!!!

(a)

	x_1	x_2	x_3	x_4	x_5
3	1	0	0	1	0
-1	1	1	0	-1	0
-4	-1	0	1	-1	0
1	1	0	0	0	1

(b)

	x_1	x_2	x_3	x_4	x_5
-1	0	0	-1	0	1
-1	1	0	0	2	-1
-1	0	0	1	1	-1
-2	-3	1	5	0	-2

(c)

	x_1	x_2	x_3	x_4	x_5	x_6
0	2	5	3	0	0	0
5	1	1	0	1	0	0
15	2	1	2	0	1	0
8	1	1	1	0	0	1

(d)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0	-1	0	2	1	-3	4	0	4
10	1	0	-1	2	1	-3	4	1
-5	0	1	1	-3	-1	2	1	-1
0	1	-1	2	-1	-1	0	0	-2

3 Duality in Linear Programming

- A nice property of Linear Programs is that every LP has a "twin" or **dual** LP — the solution of either can be constructed from the other.
- The "duality" property is mathematically ingenious.
- More to the point; it greatly improves and extends the Simplex algorithm.

3.1 Dual Linear Programs

The following pair of LPs are often called the **standard dual pair**:

$$\begin{aligned} \min \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max \quad & \mathbf{b}^\mathsf{T}\mathbf{y} \\ \text{subject to} \quad & A^\mathsf{T}\mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \geq 0. \end{aligned}$$

- I will refer to each of these LPs as the dual of the other.
- In fact I will show you that I can start with either, apply certain rules & get the other.
- I'll often refer to one of them as the **primal** and the second as the **dual**.
- Which is which depends on the context.

- Here's an example:

Example 3.1 *These two LPs are a* **standard dual pair***:*

$$\min \quad 6x_1 + 2x_2 + 3x_3 \qquad \textbf{(Primal)}$$

$$\textit{subject to} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{x} \geq 0$$

$$\max \quad 2y_1 + y_2 \qquad \textbf{(Dual)}$$

$$\textit{subject to} \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{y} \geq 0.$$

- There are some simple patterns relating this (or any) dual pair:
  - ⋆ One problem is a minimisation, the other a maximisation.
  - ⋆ Both problems are inequality constrained; the direction of the inequalities "flips" going from one to the other.
  - ⋆ The A matrix for one LP is replaced by its **transpose** A^{T} for the other.
  - ⋆ The objective function vector $\mathbf{c}$ swaps place with the "constant column" vector $\mathbf{b}$ for the other.
- From these patterns or "structural relationships" I can deduce some useful algebraic **duality relations** between the primal/dual pair of LPs.

3.1.1 Duality Relations

Here's a list of some "formulas" or **duality relations** that I can derive from the structural relationships above.

(a) If $\mathbf{x}$ is feasible for the (Primal) minimisation problem and $\mathbf{y}$ is feasible for the (Dual) maximisation problem then $\mathbf{c}^{\mathsf{T}}\mathbf{x} \geq \mathbf{b}^{\mathsf{T}}\mathbf{y}$.

(b) If one of the Primal/Dual pair has an optimal point then so does the other — **and the optimal values are equal**.

(c) If both problems are feasible then neither problem is unbounded (so both have an optimal point).

(d) If one problem is feasible but unbounded then the other problem is infeasible.

(e) It is possible for both problems to be infeasible.

I'll check these duality relations one by one.

(a)
- Assume that $\mathbf{x}$ is feasible for the minimisation problem and $\mathbf{y}$ is feasible for the maximisation problem.
- Then (make sure that you can justify each step)

$$\mathbf{c} \geq A^{\mathsf{T}}\mathbf{y} \quad \text{and} \quad \mathbf{x} \geq 0 \quad \Rightarrow \mathbf{x}^{\mathsf{T}}\mathbf{c} \geq \mathbf{x}^{\mathsf{T}}(A^{\mathsf{T}}\mathbf{y})$$

$$A\mathbf{x} \geq \mathbf{b} \quad \text{and} \quad \mathbf{y} \geq 0 \quad \Rightarrow \mathbf{y}^{\mathsf{T}}A\mathbf{x} \geq \mathbf{y}^{\mathsf{T}}\mathbf{b}.$$

- But $\mathbf{x}^{\mathsf{T}}(A^{\mathsf{T}}\mathbf{y}) = \mathbf{y}^{\mathsf{T}}(Ax)$ (same as saying $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$) so the result follows.

(b) **This is the tricky (and important) one. See below.**

(c)
- If $\mathbf{x}$ is feasible for the minimisation problem then relation (a) shows that $\mathbf{c}^\mathsf{T}\mathbf{x}$ is an upper bound on $\mathbf{b}^\mathsf{T}\mathbf{y}$ for any $\mathbf{y}$ that is feasible for the maximisation problem.
- Similarly, if $\mathbf{y}$ is feasible for the maximisation problem then, again by (a), $\mathbf{b}^\mathsf{T}\mathbf{y}$ is a lower bound on $\mathbf{c}^\mathsf{T}\mathbf{x}$ for any $\mathbf{x}$ that is feasible for the minimisation problem.
- So neither LP can be unbounded if both are feasible.
- Therefore if both problems are feasible then they must both have optimal points.

(d)

- Again using (a), if the minimisation problem is feasible but unbounded (below) then the maximisation problem cannot have any feasible point.
- This follows as if say $\bar{\mathbf{y}}$ was feasible for the maximisation problem then $\mathbf{b}^\top \bar{\mathbf{y}}$ would be a lower bound for the minimisation problem
- This contradicts the fact that the minimisation problem is unbounded.
- You should easily be able to put a similar argument together to show that the minimisation problem is infeasible if the maximisation problem is unbounded.
- Relation (d) follows.

(e) • I can show this by finding an example (a "pathological" example) of a standard dual pair where both are infeasible:

$$\begin{aligned} \min -x & \qquad \max y \\ \text{subject to } 0x \geq 1 & \qquad \text{subject to } 0y \leq -1 \\ x \geq 0 & \qquad y \geq 0. \end{aligned}$$

• Clearly both problems are infeasible.

(b) • This is the most important duality relation.

• It reads:

⋆ **If one LP has an optimal point then so does the other**

⋆ **and the optimal objective function values are equal**.

• I'll prove it in the next section.

3.2 Getting a Dual Solution From the Primal Solution

- If I solve one of the problems in a standard dual pair, the solution contains information that allows me to easily solve the other.
- To make the point, I'll look again at the pair in Example 3.1.

- Here they are again:

$$\min \quad 6x_1 + 2x_2 + 3x_3 \qquad \textbf{(Primal)}$$

$$\text{subject to} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{x} \geq 0$$

$$\max \quad 2y_1 + y_2 \qquad \textbf{(Dual)}$$

$$\text{subject to} \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{y} \geq 0.$$

- Neither problem is in standard form.
- To put the Primal (minimisation) problem in standard form ,
  - ⋆ just multiply A and $\mathbf{b}$ by -1 to reverse the direction of the inequalities
  - ⋆ then add slack variables s_1 and s_2 (say).
  - ⋆ Giving the tableau:

$P_0 =$

	x_1	x_2	x_3	s_1	s_2
0	6	2	3	0	0
-2	-1	-1	0	1	0
-1	-1	1	-1	0	1

The tableau already has the columns of the 2×2 identity matrix in place with zeros at the tops of the columns but is not in canonical form.

⋆ Using DSM, pivot on the (2,3) (Row 1, x_2 column) element of P_0 (why?) to get the tableau:

$P_1 =$

	x_1	x_2	x_3	s_1	s_2
-4	4	0	3	2	0
2	1	1	0	-1	0
-3	-2	0	-1	1	1

Still not in canonical form.

⋆ Again use DSM to pivot on the (3,2) (Row 2, x_1 column) element of P_1 (why?) to get the optimal tableau P^* :

$P^* =$

	x_1	x_2	x_3	s_1	s_2
-10	0	0	1	4	2
$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$1\frac{1}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

⋆ Reading P^*, I see that the minimisation problem has an optimal value of $\mathbf{c}^{\mathsf{T}}x^* = 10$ and and optimal point

$$x^* = \begin{bmatrix} 1\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

- Now I'll do the same procedure on the maximisation (Dual) problem.
  - ⋆ First change the problem to a minimisation problem by multiplying the objective vector $(2, 1)^T$ by -1.
  - ⋆ Now add slacks w_1, w_2 and w_3 to get the canonical form tableau:

$$D_0 = \begin{array}{|c|ccccc|}
\hline
 & y_1 & y_2 & w_1 & w_2 & w_3 \\
\hline
0 & -2 & -1 & 0 & 0 & 0 \\
\hline
6 & 1 & 1 & 1 & 0 & 0 \\
2 & 1 & -1 & 0 & 1 & 0 \\
3 & 0 & 1 & 0 & 0 & 1 \\
\hline
\end{array}$$

⋆ Two pivots with SM (check with `Pivot.m`) give the optimal tableau:

$$D^* = \begin{array}{|c|ccccc|}
\hline
 & \mathbf{y_1} & \mathbf{y_2} & \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \\
\hline
10 & 0 & 0 & 1\frac{1}{2} & \frac{1}{2} & 0 \\
\hline
2 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\
4 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\
\hline
\end{array}$$

⋆ Remember that I had to multiply the objective coefficients by -1 to change maximisation to minimisation .

⋆ So the optimal value of the maximisation problem is $\mathbf{b}^{\mathsf{T}}\mathbf{y}^* = 10$.

⋆ The optimal point is $\mathbf{y}^* = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

- I'll write the two optimal tableaux again for ease of comparison:

$P^* =$

	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$
-10	0	0	1	4	2
$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$1\frac{1}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

$D^* =$

	$\mathbf{y_1}$	$\mathbf{y_2}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$
10	0	0	$1\frac{1}{2}$	$\frac{1}{2}$	0
2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0
4	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0
1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1

- Comparing P^* and D^*:
  - ⋆ The optimal point for the minimisation problem $x^* = \begin{bmatrix} 1\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}^T$ has components that are exactly the cost coefficients of the **slack** variables w_1, w_2 and w_3 in D^*, the optimal tableau for the maximisation problem.
  - ⋆ The optimal point for the maximisation problem $\mathbf{y}^* = \begin{bmatrix} 4 & 2 \end{bmatrix}^T$ has components that are exactly the cost coefficients of the **slack** variables s_1 and s_2 in P^*, the optimal tableau for the minimisation problem.
  - ⋆ The objective function **value** entries for the two tableaux are equal and opposite as the optimal values for the original minimisation and maximisation problems are equal.
  - ⋆ I haven't yet proved that this is always true — duality relation (b) in Sec. 3.1.1.

3.2.1 Block Matrix Methods

Before the next Section I need to explain an idea that you may not have seen before; Block Matrix Algebra.

- Suppose that I have a matrix A which can be "partitioned" into rectangular pieces.
- For example the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix}$ could be written $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where $A_{11} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 4 & 5 \\ 9 & 10 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 11 & 12 & 13 \end{bmatrix}$ and $A_{22} = \begin{bmatrix} 14 & 15 \end{bmatrix}$.

- Why bother?

- Well suppose that a second matrix $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$.

- Then **provided the block matrices $A_{11}, \cdots B_{22}$ all have the right number of of rows and columns** I can write the matrix product AB as

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

- This idea will be very useful in the next Section.

- Just to make sure that you get it.
- Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where

  $A_{11} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 4 & 5 \\ 9 & 10 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 11 & 12 & 13 \end{bmatrix}$ and $A_{22} = \begin{bmatrix} 14 & 15 \end{bmatrix}$ as above.

- Let $B = \begin{bmatrix} 10 & 20 & 30 & 40 & 50 & 60 \\ 11 & 21 & 31 & 41 & 51 & 61 \\ 12 & 22 & 32 & 42 & 52 & 62 \\ 13 & 23 & 33 & 43 & 53 & 63 \\ 14 & 24 & 34 & 44 & 54 & 64 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$.

- With
$B_{11} = \begin{bmatrix} 10 & 20 \\ 11 & 21 \\ 12 & 22 \end{bmatrix}, B_{12} = \begin{bmatrix} 30 & 40 & 50 & 60 \\ 31 & 41 & 51 & 61 \\ 32 & 42 & 52 & 62 \end{bmatrix}, B_{21} = \begin{bmatrix} 13 & 23 \\ 14 & 24 \end{bmatrix}$
and $B_{22} = \begin{bmatrix} 33 & 43 & 53 & 63 \\ 34 & 44 & 54 & 64 \end{bmatrix}$, can you see that
$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$?

- This is easily checked in Matlab or Octave:

```
A11=[1 2 3;6 7 8]
A12=[4 ;9 10]
A21=[11 12 13]
A22=[14 15]
A=[A11 A12;A21 A22]
B11= [ 10 20;11 21;12 22]
B12=[30 40 50 60;    31 41 51 61;    32 42 52 62]
B21=[13 23;14 24]
B22=[ 33 43 53 63;34 44 54 64]
B=[B11 B12;B21 B22]
```

- `A*B % Do it one way`
- `[A11*B11+A12*B21 A11*B12+A12*B22;`
  `A21*B11+A22*B21 A21*B12+A22*B22] % Do it the other way`
- In both cases I get $\begin{bmatrix} 190 & 340 & 490 & 640 & 790 & 940 \\ 490 & 890 & 1290 & 1690 & 2090 & 2490 \\ 790 & 1440 & 2090 & 2740 & 3390 & 4040 \end{bmatrix}$
- OK?

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- Final points re block operations;
  - ⋆ the matrices A and B to be multiplied must be "compatible" — the number of columns in A must equal the number of rows in the second,
  - ⋆ the same must be true of the blocks in A and B to be multiplied,
  - ⋆ there is no need for A and B to both be partitioned into 4 blocks provided these rules are observed.
- For example, let A be $m \times n$ and B be $n \times p$.
  - ⋆ Let A be partitioned into k block rows and l block columns.
  - ⋆ Then B must be partitioned into l block rows and **any** number q of block columns.
  - ⋆ With the proviso that $k|m$ (k divides into m), $l|n$ and $q|p$.
  - ⋆ Draw a sketch of A and B!

3.2.2 Constructing an Optimal Dual Vector

- The dual relationships illustrated by the Example hold in general for the optimal tableaux of dual LPs.
- From the optimal tableau for either problem in a standard dual pair I can always construct the optimal solution to the other problem by using the following rule:
  - ⋆ **The optimal point for the maximisation problem has components equal to the cost coefficients of the slack variables in the optimal tableau for the minimisation problem.**
  - ⋆ **The optimal point for the minimisation problem has components equal to the cost coefficients of the slack variables in the optimal tableau for the maximisation problem.**

- To show that this rule holds in general,
  - ⋆ I'll solve the minimisation problem of the standard dual pair **algebraically** (instead of numerically as usual)
  - ⋆ then use duality relations to show that the cost coefficients for the slack variables are (as claimed) the elements of a vector that is optimal for the dual problem.
- I'll start with the standard dual pair (as previously):

$$\begin{array}{rlcrl} \min & \mathbf{c}^{\mathsf{T}}\mathbf{x} & \qquad & \max & \mathbf{b}^{\mathsf{T}}\mathbf{y} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b} & & \text{subject to} & A^{\mathsf{T}}\mathbf{y} \leq \mathbf{c} \\ & \mathbf{x} \geq 0 & & & \mathbf{y} \geq 0. \end{array}$$

- As in the worked Example, I'll introduce slack variables $\mathbf{s} \geq 0$ to reformulate the minimisation problem into standard form.
- This generates the standard form simplex tableau :

$$\mathsf{P} = \begin{array}{|c|c|c|} \hline & \mathbf{x} & \mathbf{s} \\ \hline 0 & \mathbf{c}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} \\ \textbf{-b} & \text{-}A & \mathbf{I} \\ \hline \end{array}$$

- If the minimisation problem has an optimal form vector x^* then applying the simplex algorithm must lead to an optimal tableau P^*.
- I'll write P^* as:

$$\mathsf{P}^* = \begin{array}{|c|c|c|} \hline & \mathbf{x} & \mathbf{s} \\ \hline \text{-d} & \mathbf{u}^{\mathsf{T}} & \mathbf{v}^{\mathsf{T}} \\ \mathbf{b}^* & D & B \\ \hline \end{array}$$

- I need to show that the vector ν of cost coefficients for the (Primal) slack variables is precisely the optimal point for the (Dual) maximisation problem.

- As the optimal tableau P^* was obtained from P by a succession of pivots, there must be a **pivot matrix** Q such that $P^* = QP$.
  - ⋆ A pivot matrix is just the result of **doing the succession of pivots on the $(m+1) \times (m+1)$ identity matrix** I_{m+1}.
  - ⋆ The first column of a pivot matrix Q is always the first column of the identity matrix — as the objective function row of a tableau is never the pivot row so row ops applied to I_{m+1} don't affect its first column! (Multiples of the 1 in the TLC don't get added to the 0's below it.)
  - ⋆ **The remaining m columns of Q are just the right-most m columns of the tableau P^* — corresponding to the m slack variables s.**

- To see this write

$$\mathsf{P} = \begin{bmatrix} \mathsf{0} & \mathbf{c}^\mathsf{T} & \mathbf{0}^\mathsf{T} \\ -\mathbf{b} & -\mathsf{A} & \mathrm{I} \end{bmatrix} = \begin{bmatrix} \mathbf{p}^\mathsf{T} & \mathbf{0}^\mathsf{T} \\ \hat{\mathsf{P}} & \mathrm{I} \end{bmatrix}.$$

- So $\mathbf{p}^\mathsf{T} = \begin{bmatrix} \mathsf{0} & \mathbf{c}^\mathsf{T} \end{bmatrix}$ and $\hat{\mathsf{P}} = \begin{bmatrix} -\mathbf{b} & -\mathsf{A} \end{bmatrix}$.

- Also take $\mathrm{Q} = \begin{bmatrix} \mathsf{1} & \mathsf{q}^\mathsf{T} \\ \mathbf{0} & \hat{\mathsf{Q}} \end{bmatrix}$.

- Then $\mathrm{QP} = \begin{bmatrix} \mathsf{1} & \mathsf{q}^\mathsf{T} \\ \mathbf{0} & \hat{\mathsf{Q}} \end{bmatrix} \begin{bmatrix} \mathbf{p}^\mathsf{T} & \mathbf{0}^\mathsf{T} \\ \hat{\mathsf{P}} & \mathrm{I} \end{bmatrix} = \begin{bmatrix} \text{stuff} & \mathsf{q}^\mathsf{T} \\ \text{more stuff} & \hat{\mathsf{Q}} \end{bmatrix}$

- So, comparing the right-hand m columns of QP and P^*; as claimed, the pivot matrix Q can be written $Q = \begin{bmatrix} 1 & \mathbf{v}^T \\ \mathbf{0} & B \end{bmatrix}$ where $\begin{bmatrix} \mathbf{v}^T \\ B \end{bmatrix}$ are the right-hand m columns of P^*.

- The gory details are
  - ⋆ Write $Q = \begin{bmatrix} 1 & q^T \\ \mathbf{0} & \hat{Q} \end{bmatrix}$.
  - ⋆ Then $QP = \begin{bmatrix} 1 & q^T \\ \mathbf{0} & \hat{Q} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{c}^T & \mathbf{0}^T \\ -\mathbf{b} & -A & I \end{bmatrix} = \begin{bmatrix} -q^T b & c^T - q^T A & q^T \\ -\hat{Q} b & -\hat{Q} A & \hat{Q} \end{bmatrix} = P^* = \begin{bmatrix} -d & \mathbf{u}^T & \mathbf{v}^T \\ \mathbf{b}^* & D & B \end{bmatrix}$.
  - ⋆ The conclusion is the same; $\mathbf{q}^{\mathbf{T}} = \mathbf{v}^{\mathbf{T}}$ and $\hat{Q} = B$.
- So the right-hand m columns of the $(m+1) \times (m+1)$ matrix Q are just the the right-hand m columns of P^*, the columns corresponding to the slack variables s.

- Summarising, the pivot matrix Q that transforms the opening tableau P into the optimal tableau P^* is:

$$Q = \left[\begin{array}{c|c} 1 & \mathbf{v}^T \\ \hline \mathbf{0} & B \end{array}\right]$$

- For the minimisation problem of the numerical Example above,

$$P^* = \begin{array}{|c|ccccc|} \hline & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{s_1} & \mathbf{s_2} \\ \hline -10 & 0 & 0 & 1 & 4 & 2 \\ \hline 1\frac{1}{2} & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & \mathbf{x} & \mathbf{s} \\ \hline -d & \mathbf{u}^T & \mathbf{v}^T \\ \mathbf{b}^* & D & B \\ \hline \end{array}$$

- So (for the numerical Example)

$$Q = \left[\begin{array}{c|c} 1 & \mathbf{v}^{\mathsf{T}} \\ \hline \mathbf{0} & B \end{array}\right] = \left[\begin{array}{c|cc} 1 & 4 & 2 \\ \hline 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] = \begin{bmatrix} 1 & 4 & 2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- Check that $QP = P^*$.
- Now, using the general form for the pivot matrix Q to compute the elements of P^* in terms of the elements of P, the matrix equation $P^* = QP$ translates to:

$$P^* = \left[\begin{array}{c|c} 1 & \mathbf{v}^{\mathsf{T}} \\ \hline \mathbf{0} & B \end{array}\right] \begin{array}{|c|c|c|} \hline 0 & \mathbf{c}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} \\ \hline -\mathbf{b} & -A & \mathbf{I} \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline -\mathbf{v}^{\mathsf{T}}\mathbf{b} & (\mathbf{c}^{\mathsf{T}} - \mathbf{v}^{\mathsf{T}}A) & \mathbf{v}^{\mathsf{T}} \\ \hline -B\mathbf{b} & -BA & B \\ \hline \end{array}$$

- The tableau P^* is in optimal form so
  - ⋆ $\mathbf{v}^T \geq 0$ and
  - ⋆ $\mathbf{u}^T = \mathbf{c}^T - \mathbf{v}^T A \geq 0$,
  - ⋆ which means that $A^T\mathbf{v} \leq \mathbf{c}$
- So $\mathbf{v}$ is feasible for the dual (maximisation) problem.
- Also $-\mathbf{d} = -\mathbf{v}^T\mathbf{b}$ so $\mathbf{d} = \mathbf{v}^T\mathbf{b}$.
- For some x, (say $\mathbf{x}^*$), $\mathbf{d} = \mathbf{c}^T x^*$ is the optimal value for the primal (minimisation) problem.
- Taking $x = x^*$ in duality relation (a) (Slide 199) I have that $\mathbf{d} \equiv \mathbf{c}^T x^* \geq b^T y$ for all dual feasible y.
- I know that $\mathbf{v}$ is feasible for the dual (maximisation) problem.
- So as $\mathbf{d} = \mathbf{v}^T\mathbf{b}$, the vector $\mathbf{v}$ must be dual optimal as it "attains the maximum".

- I've shown that an optimal point for the maximisation problem can be constructed from the coefficients for the slack variables in the optimal tableau for the minimisation problem.
- And that the optimal values of the two LPs are equal.
- It is easy (see the Exercises for this Chapter) to put a similar argument together that shows that an optimal point for the minimisation problem has its components equal to the cost coefficients of the slack variables in the optimal tableau for the maximisation problem.
- I have proved duality relation (b) on Slide 199.

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3.3 Economic Interpretation of Dual Variables

- Dual variables often have an important economic interpretation.
- I'll illustrate this with a simple resource allocation problem.

Example 3.2 (Widgit Corp.) *A firm, Widgit Corp :*

- ⋆ *manufactures four products* P_1–P_4
- ⋆ *uses three resources,* R_1–R_3
- ⋆ *needs resource inputs and generates revenue according to:*

	P_1	P_2	P_3	P_4	*Available*
R_1	2	1	1	1	30
R_2	1	0	2	1	15
R_3	1	1	1	0	5
Revenue	6	1	4	5	

- For example, one unit of product P_3 sells for €4 and consumes 1, 2 and 1 unit of the resources R_1, R_2 and R_3 respectively.
- The problem is to choose the production plan (amount x_i of each product P_i to produce) in order to maximise the revenue subject to available resources.
- I need to solve the LP:

$$\begin{aligned}
\max \quad & 6x_1 + x_2 + 4x_3 + 5x_4 \\
\text{subject to} \quad & \\
& 2x_1 + x_2 + x_3 + x_4 && \leq 30 \\
& x_1 + 0x_2 + 2x_3 + x_4 && \leq 15 \\
& x_1 + x_2 + x_3 + 0x_4 && \leq 5 \\
& x_1, x_2, x_2, x_4 \geq 0.
\end{aligned}$$

- This LP takes the form of the maximisation problem in the standard dual pair.
- When reformulated into standard form, I have the canonical form tableau :

$$P = \begin{array}{|c|ccccccc|}
\hline
 & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_2} & \mathbf{s_3} \\
\hline
0 & -6 & -1 & -4 & -5 & 0 & 0 & 0 \\
\hline
30 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
15 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\
5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

where s_1, s_2 and s_3 are the slacks associated with the corresponding resources.

- An optimal tableau for the problem is:

$$P^* = \begin{array}{|c|ccccccc|} \hline & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_2} & \mathbf{s_3} \\ \hline 80 & 0 & 0 & 7 & 0 & 0 & 5 & 1 \\ \hline 10 & 0 & 0 & -2 & 0 & 1 & -1 & -1 \\ 10 & 0 & -1 & 1 & 1 & 0 & 1 & -1 \\ 5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

- So the optimal production plan for Widgit Corp is to produce 5 units of P_1, 10 units of P_4 and none of P_2 or P_3.
- This plan yields the maximum revenue of €80. (No minus sign as this is a max problem.)
- The optimal values of the slack variables s_1, s_2 and s_3 tell me that 10 units of resource R_1 are unused in the optimal production plan and that all of resources R_2 & R_3 are used.

- Now suppose that, before the production run started, someone offered to buy one unit of resource R_2.
- The firm would get some income from the sale (I don't know how much as that information wasn't given.)
- But by reducing the amount of of resource R_2 available,
  - ⋆ the optimal production plan may be changed
  - ⋆ and the revenue from production may be reduced.
  - ⋆ So a reasonable question is:
    What is the minimum price that should be charged for that one "marginal" unit of resource R_2 if the total revenue is to remain $\geq$ €80?

- The slack variable s_2 is zero in the optimal tableau P^* so **all** of resource R_2 is used in the optimal production plan x^* .
- This means that selling one unit of R_2 prior to production has the same effect as requiring that one unit be left over after the production run.
- So $s_2 = 1$ in the new optimal solution.
- I can work out the effect on the optimal solution of requiring that $s_2 = 1$ by simply letting $s_2 = 1$ in the optimal tableau P^*.

- I can write the equations represented by P^* so that s_2 appears on the LHS along with the constant column, giving the new tableau P':

$$
P' = \begin{array}{|c|cccccc|}
\hline
 & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_3} \\
\hline
80 - 5s_2 & 0 & 0 & 7 & 0 & 0 & 1 \\
\hline
10 + s_2 & 0 & 0 & -2 & 0 & 1 & -1 \\
10 - s_2 & 0 & -1 & 1 & 1 & 0 & -1 \\
5 & 1 & 1 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

- As long as $10 + s_2 \geq 0$ and $10 - s_2 \geq 0$, $(-10 \leq s_2 \leq 10)$ the new tableau is in canonical form and revenue is $80 - 5s_2$.
- So if **total** revenue is to remain at least €80 then I need:

  (revenue from production)+(revenue from sales of resources) ≥ 80.

- This translates into

$$(80 - 5s_2) + s_2 \cdot (\textbf{selling price/unit}) \geq 80.$$

- Solving for **selling price/unit** (remembering that $s_2 \geq 0$) gives

$$\text{selling price/unit} \geq \text{€}5.$$

- This minimum price of €5/unit for resource R_2 is called the **shadow price** of the resource.

- What about the other resources?
  - ⋆ If I combined the s_1 column with the constant column I wouldn't have a canonical form tableau as the s_1 column is a column of the identity matrix.
  - ⋆ But as $s_1 = 10$ in the optimal solution x^* , I know that there are 10 units of resource R_1 left over after production, not used in the optimal production plan.
  - ⋆ So Widgit Corp could give units of resource R_1 away for nothing without affecting total revenue, provided $s_1 \leq 10$.
  - ⋆ So the shadow price of R_1 is zero.
- You should check that the shadow price of resource R_3 is €1.

- I have found that the shadow prices for the three resources are $(0, 5, 1)^{\mathsf{T}}$.
- In the previous Section I showed that the optimal dual vector can be read off the optimal tableau T^* as the cost coefficients of the slack variables.
- This vector, $\mathbf{y}^* = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$ has elements that are exactly the shadow prices for the three resources.

- This correspondance between shadow prices and the slack variable cost coefficients holds in general:

  The optimal vector for the dual of a resource allocation problem has components that are the shadow prices of the resources.

- Another way to "see" this result is to realise that the cost coefficients in the optimal tableau for the **slack** variables (which I know are the optimal dual vector components) measure the effect on the objective of increasing the corresponding **slack** variable by one unit from zero.

- So as a standard form tableau represents a minimisation problem, the cost coefficients for the **slack** variables tell me by how much revenue is **reduced** if I reduce the availability of the corresponding resource by one unit.

- So the **correspondance** above follows.

3.4 Finding the Dual of Any LP

- If an LP has the form of one or the other of the "twin" problems in a standard dual pair, then I can simply write down the other problem as its dual.
- The duality relations then hold automatically.
- What if the LP isn't in the form of one of the problems in a standard dual pair?
- Just reformulate the LP algebraically so that it **is** in the form of one of the problems in a standard dual pair.
- Then simply write down the dual problem.

- This procedure works because it is **always** possible to reformulate any LP so that it is in the form of one of the problems in a standard dual pair.
- I'll show how this can be done in the next Subsection.

3.4.1 Dual of an LP in Standard Form

- A standard form LP takes the form

$$\begin{aligned} \min \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- Some tricks are needed to transform it into one of the problems in a standard dual pair.
- First, $A\mathbf{x} = \mathbf{b}$ can be replaced by $A\mathbf{x} \leq \mathbf{b}$ **and** $A\mathbf{x} \geq \mathbf{b}$.
- Or by $-A\mathbf{x} \geq -\mathbf{b}$ **and** $A\mathbf{x} \geq \mathbf{b}$.

- So the original standard form LP can be written:

$$\begin{aligned} \min \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \geq \mathbf{b} \\ & -A\mathbf{x} \geq -\mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- Which in turn can be written with a single matrix inequality:

$$\begin{aligned} \min \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & \begin{bmatrix} A \\ -A \end{bmatrix} \mathbf{x} \geq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- This reformulated problem **does** take the form of one of the problems in a standard dual pair.

- So I can write its dual (setting $\mathbf{y}^\mathsf{T} = [\mathbf{u}^\mathsf{T}\mathbf{v}^\mathsf{T}]$ — why?):

$$\begin{aligned} \max \quad & [\mathbf{b}^\mathsf{T}, -\mathbf{b}^\mathsf{T}] \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \\ \text{subject to} \quad & \begin{bmatrix} A^\mathsf{T} & -A^\mathsf{T} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \leq \mathbf{c} \\ & \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \geq \mathbf{0}. \end{aligned}$$

- This can be unpacked into:

$$\begin{aligned} \max \quad & \mathbf{b}^\mathsf{T}(\mathbf{u} - \mathbf{v}) \\ \text{subject to} \quad & A^\mathsf{T}(\mathbf{u} - \mathbf{v}) \leq \mathbf{c} \\ & \mathbf{u}, \mathbf{v} \geq \mathbf{0}. \end{aligned}$$

- Finally!
- The difference of two non-negative variables/vectors can be identified as a **free** variable/vector, say $\mathbf{y} = \mathbf{u} - \mathbf{v}$.
- So I can say that the dual of a standard form LP is:

$$\begin{aligned} \max \quad & \mathbf{b}^{\mathsf{T}}\mathbf{y} \\ \text{subject to} \quad & A^{\mathsf{T}}\mathbf{y} \le \mathbf{c} \\ & \mathbf{y} \quad \text{free.} \end{aligned} \tag{3.1}$$

3.4.2 Trickier Examples

- Sometimes a little more work is needed.
- Try this example:

$$\begin{aligned} \max \quad & \mathbf{a}^\mathsf{T}\mathbf{y} + \mathbf{b}^\mathsf{T}\mathbf{w} \\ \text{subject to} \quad & A\mathbf{y} + B\mathbf{w} \geq \mathbf{c} \\ & D\mathbf{w} \leq \mathbf{d} \\ & \mathbf{w} \geq \mathbf{0}, \mathbf{y} \quad \text{free.} \end{aligned}$$

- A very nice result (which I leave as an exercise) is that:

  Exercise 3.1 **The dual of the dual of any LP is the original LP!**

- So start with one of the standard dual pair;

$$
\begin{array}{rlcrl}
\min & \mathbf{c}^{\mathsf{T}}\mathbf{x} & \qquad & \max & \mathbf{b}^{\mathsf{T}}\mathbf{y} \\
\text{subject to} & A\mathbf{x} \geq \mathbf{b} & & \text{subject to} & A^{\mathsf{T}}\mathbf{y} \leq \mathbf{c} \\
 & \mathbf{x} \geq 0 & & & \mathbf{y} \geq 0.
\end{array}
$$

- Find its dual (just read across).
- Then (not so easy) reformulate this LP into the form of the starting LP!
- Read across again.
- Finally, unpack the result — you will have the original LP!

3.5 The Dual Simplex Method

- In a lot of LP problems (including sensitivity analysis — to be discussed in the last Chapter) the following situation can arise.
- Having solved the LP and found an optimal tableau, a change in the problem data or maybe a new constraint results in a new tableau
  - ⋆ that still has non-negative cost coefficients and a set of basic variables (so still optimal)
  - ⋆ but the constant column has some negative entries (so infeasible)
  - ⋆ I showed you in Section 2.7 that the DSM can be used to transform such a tableau into canonical form — if that is possible.
  - ⋆ So I can use the DSM to find an optimal solution to the new LP with the extra constraints.

Dual Simplex Example I'm now going to derive the DSM (which we have already used as a tool in Section 2.7). First I'll illustrate the technique with an example.

Example 3.3 (Dual Simplex Example) *Suppose that the Simplex Method has been used to solve an LP and that the following optimal tableau has been found:*

	x_1	x_2	x_3	x_4	x_5
70	0	2	1	0	4
50	0	1	-2	1	-2
5	1	1	1	0	-1

- Now suppose that the optimal point represented by this tableau ($(x_1, x_2, x_3, x_4, x_5)^T = (5, 0, 0, 50, 0)^T$) is not acceptable and I add the extra constraint $x_2 + x_3 \geq 10$.

- Introducing a slack variable x_6 for this new constraint,
  - ⋆ I could write it as $x_2 + x_3 - x_6 = 10$
  - ⋆ I need to introduce an extra column of the Identity matrix in the tableau corresponding to the extra constraint row
  - ⋆ So I write the new constraint as $-x_2 - x_3 + x_6 = -10$.
- So the extended problem has the following Standard Form tableau:

	x_1	x_2	x_3	x_4	x_5	x_6
70	0	2	1	0	4	0
50	0	1	−2	1	−2	0
5	1	1	1	0	−1	0
−10	0	−1	−1	0	0	1

- The tableau is no longer Canonical as the LHS Column has a negative element and in fact does not represent a feasible point as $b_3 = -10$.
- However it does still have all non-negative cost coefficients – the mark of an optimal point.
- So the tableau represents an "infeasible but optimal" point (strictly speaking not possible but useful shorthand).

- However, if a tableau has non-negative **costs** and a set of basic variables (columns from the identity matrix) then
  - ⋆ While keeping the cost coefficients non-negative
  - ⋆ I can pivot to either
    - ∗ make the LH Column non-negative (so I have an optimal tableau) or
    - ∗ show that the problem is infeasible.
- To derive a set of "Dual Simplex Pivot Rules", consider the following primal/dual pair with $c \geq 0$ as in Example 3.3:

$$\begin{array}{rlcrl} & \min c^T x & \qquad & & \max -b^T y \\ \text{subject to} & Ax \leq b & & \text{subject to} & -A^T x \leq c \\ & x \geq 0 & & & y \geq 0. \end{array}$$

(It is easy to show that this is a dual pair. See the definition in Section 3.1.) N.B. $\max -b^T y \equiv -\min b^T y$.

- If I introduce non-negative "slack" vectors $\mathbf{u}$ and $\mathbf{s}$ to the primal & dual problem respectively I can represent the two problems as Standard Form Simplex tableaux:

$P =$

	$\mathbf{x}$	$\mathbf{u}$
0	$\mathbf{c}^\mathsf{T}$	$\mathbf{0}^\mathsf{T}$
$\mathbf{b}$	A	I

and $D =$

	$\mathbf{y}$	$\mathbf{s}$
0	$\mathbf{b}^\mathsf{T}$	$\mathbf{0}^\mathsf{T}$
$\mathbf{c}$	$-A^\mathsf{T}$	I

- N.B. **as I am restricting $\mathbf{c} \geq \mathbf{0}$**, the primal tableau P is optimal, but possibly not feasible.

- If $\mathbf{b} \geq 0$ then P is feasible & optimal.

- The dual tableau D is **already** in canonical form as $\mathbf{c} \geq \mathbf{0}$.

- To transform D to a tableau with $\mathbf{b} \geq 0$ while keeping $\mathbf{c} \geq \mathbf{0}$, just pivot using the Simplex Pivot Rules!

- In the dual tableau D, I know that pivoting using the Simplex Pivot Rules will end in either an optimal or an unbounded tableau.
- Because these rules arise from applying the Simplex Algorithm to the Dual tableau, this pivoting method is called the **Dual Simplex Method**.
- On the next Slide I will write down explicitly what these rules are.

- First note that (because $-A^T$ appears in D, not A) the column in D under b_i has entries $-a^T_{ji} = -a_{ij}$ for $j = 1, \ldots, n$ —
  - ⋆ the i^{th} **column** of $-A^T$
  - ⋆ the i^{th} **row** of $-A$.
- So the minimum ratio row j for the b_i column is given by:

$$\min_j \left\{ \frac{c_j}{-a_{ij}} \text{such that } -a_{ij} > 0 \right\}$$

- or (obviously) equivalently by:

$$\max_j \left\{ \frac{c_j}{a_{ij}} \text{such that } a_{ij} < 0 \right\}$$

- So applying the Simplex Algorithm to the Dual tableau D, I get the **Dual Simplex Method** :

Algorithm 3.1 (Dual Simplex Method) *(Start with a tableau s.t.* $\mathbf{c} \geq \mathbf{0}$*.)*

while *NOT finished* ***do***

if $b_i \geq 0$ *for all* i ***then***

STOP *(Tableau is optimal.)*

else

Select i *s.t.* $b_i < 0$.

end if

if $a_{ij} \geq 0$ *for all* $j = 1, \ldots, n$ ***then***

STOP *(Dual unbounded* $\equiv$ *Primal infeasible.)*

end if

Select a row k *such that*

$\frac{c_k}{a_{ik}} = \max_j \left\{ \frac{c_j}{a_{ij}} \text{such that } a_{ij} < 0 \right\}$ *(*k *attains max.)*

Pivot on a_{ik}*. (Divide Row* k *across by* a_{ik} *and add ... multiples of Row* k *to the rows above & below.)*

end while

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- By writing the rules in this form I can see that if I perform pivots on the primal tableau **using these rules** the pivots maintain non-negativity of the cost coefficients $\mathbf{c}$
  - ⋆ just as SM maintains non-negativity of Column 0 in a canonical form tableau

  and either achieve non-negativity of the LH column $\mathbf{b}$ or end with an infeasible tableau.
- I illustrate with Example 3.5 above — labelling the tableaux P & D.
- To make the example easier to follow I first rearrange the columns of P so that the columns of the Identity matrix are on the right.

Shuffle the columns of

	x_1	x_2	x_3	x_4	x_5	x_6
70	0	2	1	0	4	0
50	0	1	−2	1	−2	0
5	1	1	1	0	−1	0
−10	0	−1	−1	0	0	1

to give P=

	x_2	x_3	x_5	x_4	x_1	x_6
70	2	1	4	0	0	0
50	1	−2	−2	1	0	0
5	1	1	−1	0	1	0
−10	−1	−1	0	0	0	1

- Writing P & D together:

P=

	x_2	x_3	x_5	x_4	x_1	x_6
70	2	1	4	0	0	0
50	1	−2	−2	1	0	0
5	1	1	−1	0	1	0
−10	−1	**−1**	0	0	0	1

and D=

	y_1	y_2	y_3	y_4	y_5	y_6
−70	50	5	−10	0	0	0
2	−1	−1	1	1	0	0
1	2	−1	**1**	0	1	0
4	2	1	0	0	0	1

- The dual simplex pivot rules select the **RED** pivot position in D.
- More important, applying these rules gives the **RED** pivot position in P.

- If I perform these pivots in P & D I find (again shuffling the columns of P & D so that the identity is on the right):

P=

	x_2	x_6	x_5	x_4	x_1	x_3
60	1	1	4	0	0	0
70	3	−2	−2	1	0	0
−5	0	1	−1	0	1	0
10	1	−1	0	0	0	1

and D=

	y_1	y_2	y_5	y_4	y_3	y_6
−60	70	−5	10	0	0	0
1	−3	0	−1	1	0	0
1	2	−1	1	0	1	0
4	2	1	0	0	0	1

- Finally, I perform these pivots in P & D and I find (again shuffling the columns of P & D so that the identity is on the right) **both tableaux are optimal**:

P=

	x_2	x_6	x_1	x_4	x_5	x_3
40	1	5	4	0	0	0
80	3	−4	−2	1	0	0
5	0	−1	−1	0	1	0
10	1	−1	0	0	0	1

and D=

	y_1	y_6	y_5	y_4	y_3	y_2
−40	80	5	10	0	0	0
1	−3	0	−1	1	0	0
5	4	1	1	0	1	0
4	2	1	0	0	0	1

- Of course in practice I **only** pivot on P using the Dual Simplex Algorithm above.
- The sequence of P/D (Primal/Dual) tableau pairs does illustrate the fact that pivoting with the Simplex Method on the Dual tableau D is equivalent to pivoting with the Dual Simplex Method on the Primal tableau P!
- In practice I also don't shuffle the columns so that the identity I appears on the right
  - ⋆ I did it here to make it easier to see the relationship between P & T.

One Last Dual Simplex Example

- The following example has non-negative ("optimal") cost coefficients and a set of basic variables.
- In this example I will not rearrange the columns so that the Identity columns appear on the right — there is no need in general as noted above.
- The following tableau would be optimal except for the fact that there are two rows with $b_i < 0$.
- I select the first row (or the last row) as the pivot row; $i = 1$.

	x_1	x_2	x_3	x_4	x_5	x_6
-10	0	0	3	1	2	0
-5	1	0	-1	0	-1	0
2	0	0	2	3	0	1
-7	0	1	2	-1	-1	0

- Now choose a column using the Rule in Line 10 of the Algorithm, the $x_3(j = 3)$ and $x_5(j = 5)$ colums both have negative a_{ij} but $\frac{2}{-1} > \frac{3}{-1}$ so I choose the x_5 column.
- The pivot element is shown in **RED** on the previous Slide.
- Pivoting gives:

	x_1	x_2	x_3	x_4	x_5	x_6
-20	2	0	1	1	0	0
5	-1	0	1	0	1	0
2	0	0	2	3	0	1
-2	-1	1	3	-1	0	0

- Choose Row 3 and the x_4 column as $\frac{1}{-1} > \frac{2}{-1}$.

-

	x_1	x_2	x_3	x_4	x_5	x_6
-22	1	1	4	0	0	0
5	-1	0	1	0	1	0
-4	$\mathbf{-3}$	3	11	0	0	1
2	1	-1	-3	1	0	0

- Choose Row 2 and the x_1 column as -3 is the only negative element in Row 2.

-

	x_1	x_2	x_3	x_4	x_5	x_6
$-70/3$	0	2	$23/3$	0	0	$1/3$
$19/3$	0	-1	$-8/3$	0	1	$-1/3$
$4/3$	1	-1	$-11/3$	0	0	$-1/3$
$2/3$	0	0	$2/3$	1	0	$1/3$

Optimal!

Exercise 3.2 *Work through the last Example, taking Row 3 rather than Row 1 as the pivot row.*

3.5.1 But Why Does DSM Work?

I mean (other than by looking at examples) how do I know that applying SM to the dual tableau D gives a tableau which is the dual of the tableau that I get if I apply DSM to the primal tableau P?

In other words how do I know that I can use DSM on the primal tableau P — much more convenient than working with both P & D?

The simplest way to "see" this is to:

- Write down a grid with entries for D in rows 0, r (the pivot row) and i (an arbitrary row with $i \neq r$) and in columns 0, c (the pivot column) and k (an arbitrary column with $k \neq c$).
- For example the entry in row i and column k is $-a_{ki}$.
- Then apply an SM pivot to the entry in row r and column c of this tableau.

Now do the same procedure for P noting (this is the slightly confusing part) that the entry in row k and column i of P is a_{ki}:

- Apply a DSM pivot on the entry in row c and column r (a_{cr}).
- Check that the entry in row k and column i of P after the DSM pivot is exactly -1 times the entry in row i and column k of D after the SM pivot.

- There are some special cases that need to be examined
  - ⋆ For example when column k of D is basic (a column from the identity matrix with zero at the top.
  - ⋆ If the 1 is in row r then the column becomes non-basic after the pivot.
  - ⋆ So at each iteration one basic column becomes non-basic and vice versa.
- But the result is as claimed, applying the DSM to P is equivalent to applying the SM to D.
- I've scanned in a handwritten version of the two "grids" for P & D — you'll find them at `http://jkcray.maths.ul.ie/ms4303/PvsD.pdf`.

3.6 Another Application of the Dual Simplex Method

Now that I know where the DSM comes from, let's look again at how it may be used to transform a Simplex tableau that "has the m identity columns appearing as columns with the corresponding cost coefficients equal to zero" into canonical form.

I'll repeat the example that I used in Section 2.7:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
81	0	0	0	2	14	-42	13	0	-12
-55	0	0	0	0	-5	25	-6	1	5
-4	1	0	0	-1	0	10	0	0	1
2	0	1	0	1	-1	1	-1	0	0
1	0	0	1	-1	2	-11	2	0	-1

The tableau has the "m identity columns appearing as columns with the corresponding cost coefficients equal to zero" property but is not in canonical form as some of the constant column entries are negative.

- The DSM can be applied provided I exclude columns with negative cost coefficient (the x_6 and x_9 columns).
- So apply DSM,
  selecting Row 2 (first constraint row) and Column 8 (x_7 column).

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
-299/6	0	0	0	2	19/6	73/6	0	13/6	-7/6
55/6	0	0	0	0	5/6	-25/6	1	-1/6	-5/6
-4	1	0	0	-1	0	10	0	0	1
67/6	0	1	0	1	-1/6	-19/6	0	-1/6	-5/6
-52/3	0	0	1	-1	1/3	-8/3	0	1/3	2/3

- Now apply DSM, pivoting on Row 5 & the x_4 column (Column 5).

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
-437/6	0	0	2	0	23/6	41/6	0	17/6	1/6
55/6	0	0	0	0	5/6	-25/6	1	-1/6	-5/6
40/3	1	0	-1	0	-1/3	38/3	0	-1/3	1/3
-37/6	0	1	1	0	1/6	-35/6	0	1/6	-1/6
52/3	0	0	-1	1	-1/3	8/3	0	-1/3	-2/3

- And finally, apply DSM, pivoting on Row 4 & Column 10.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
-79	0	1	3	0	4	1	0	3	0
40	0	-5	-5	0	0	25	1	-1	0
1	1	2	1	0	0	1	0	0	0
37	0	-6	-6	0	-1	35	0	-1	1
42	0	-4	-5	1	-1	26	0	-1	0

3.7 Exercises for Chapter 3

1. (a) Write the dual of the T&C Corp problem Example 1.1. (The algebraic description is on Slide 23.)

   (b) Find an initial standard form tableau for the dual and pivot to optimal form.

   (c) From the optimal tableau for the dual, read off an optimal vector for the primal.

   (d) Suppose that a customer offered to buy one unit of the 12.5 units of wood available for production. What is the smallest amount that the Company should accept for that unit of wood?

(e) Write down an initial standard form tableau for the T&C Corp problem and pivot to optimal form. Read off the optimal vector for the dual from the tableau and check that it agrees with the one found in part (b).

(f) What economic interpretation can be given to the optimal value of the dual variable y_1?

2. Find the dual of the following problems:

(a)

$$\max x_1 + x_2$$

subject to

$$\begin{aligned} -2x_1 + 2x_2 &\leq 1 \\ 16x_1 - 14x_2 &\leq 7 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(b)

$$\min 2x_1 - 7x_2 - 3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 5 \\ 2x_1 + 0x_2 + x_3 &\leq 10 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

(c)

$$\min x_1 - x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\geq 4 \\ 0x_1 + x_2 &\leq 4 \\ 3x_1 - 2x_2 &\leq 0 \\ x_1, x_2 &\geq 0. \end{aligned}$$

3. Consider the following LP:

$$\max x_1 + 2x_2 + 5x_3 + 4x_4$$

subject to

$$x_1 + x_2 + x_3 + x_4 \leq 10$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

(a) Write down the dual of this LP.

(b) Write down the solution simply by examining the dual — no algebra needed.

(c) Write the simplex tableau for the LP & pivot to solve it

(d) Write the simplex tableau for the dual LP & pivot to solve it.

(e) Interpret the results!

4. (a) Find the dual of the refinery problem Example 1.4.
   (b) Solve the dual problem.
   (c) What are the shadow prices for each grade of crude oil?

5. Suppose that a resource allocation problem $\max \mathbf{c}^T\mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ with $\mathbf{x} \geq 0$ has the following initial canonical form tableau P when slack variables have been added to each of the three resource constraints.

$P =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
0	-2	-5	-3	-7	0	0	0
50	1	1	0	3	1	0	0
150	2	1	2	1	0	1	0
80	1	1	1	4	0	0	1

After pivoting with the Simplex algorithm , the optimal tableau P* is:

$$
\mathsf{P}^* = \begin{array}{|c|ccccccc|}
\hline
340 & 3 & 0 & 0 & 11 & 2 & 0 & 3 \\
\hline
50 & 1 & 1 & 0 & 3 & 1 & 0 & 0 \\
40 & 1 & 0 & 0 & -4 & 1 & 1 & -2 \\
30 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \\
\hline
\end{array}
$$

(a) Prior to production, what is the least price that the Company should accept for one unit of Resource 1 so that the Company's total revenue (production plus sale of resources) stays at €340?

(b) Answer the same question for Resources 2 & 3.

(c) Up to what amount of Resource 1 sold is the per-unit price that you found in part (a) valid?

Hint: use the optimal tableau to write each basic variable in terms of s_1. By how much can s_1 increase from zero before one of the basic variables becomes negative?

6. Find the dual of the following LPs:

(a)

$$\min \mathbf{c}^\mathsf{T}\mathbf{x} + \mathbf{a}^\mathsf{T}\mathbf{y}$$

subject to

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ D\mathbf{x} + B\mathbf{y} &\geq \mathbf{d} \\ \mathbf{x} \geq 0, \mathbf{y} &\text{ free.} \end{aligned}$$

(b)

$$
\begin{aligned}
&\min \mathbf{c}^{\mathsf{T}}\mathbf{x} \\
&\text{subject to} \\
&\quad A\mathbf{x} = \mathbf{b} \\
&\quad B\mathbf{x} \leq \mathbf{a} \\
&\quad \mathbf{x} \geq 0.
\end{aligned}
$$

(c)

$$
\begin{aligned}
&\min \mathbf{c}^{\mathsf{T}}\mathbf{w} + \mathbf{a}^{\mathsf{T}}\mathbf{v} \\
&\text{subject to} \\
&\quad A\mathbf{w} + B\mathbf{v} = \mathbf{b} \\
&\quad B\mathbf{x} \leq \mathbf{a} \\
&\quad \mathbf{w} \geq 0, \mathbf{v} \text{ free.}
\end{aligned}
$$

7. Solve the following LP using the Dual Simplex method:

$$\min 3x_1 + x_2$$

subject to

$$x_1 - x_2 \leq -1$$

$$-x_1 - x_2 \leq -4$$

$$x_1 + x_2 \leq 1$$

$$x_1, x_2 \geq 0.$$

8. Suppose that the following optimal tableau is obtained by solving a LP in standard form.

	x_1	x_2	x_3	x_4	x_5	x_6
-10	0	0	5	4	0	3
4	1	0	-1	2	0	-1
2	0	0	4	1	1	2
5	0	1	1	1	0	-1

Due to a management error, the constraint $x_1 + x_2 + x_3 \leq 8$ was not included in the original model. Add this constraint to the above tableau and pivot to get a new optimal form tableau. (Use `Pivot.m`.)

4 The Transportation Problem

- Many LP models have a special structure that allows the simplex algorithm to be implemented very efficiently.
- This allows bigger problems to be solved that might otherwise be impossible.
- In this Chapter I'll show how the simplex algorithm can be specialised for Transportation Problems.

4.1 Introduction to Transportation Problems

- This was one of the earliest examples where the special structure of a problem was exploited to simplify the Simplex algorithm.
- In the **transportation problem** units of a single product are to be shipped from m sources to n destinations where:
  - ⋆ a_i is the "supply", the number of units available at source i
  - ⋆ b_j is the number of units required (the "demand") at destination j
  - ⋆ c_{ij} is the cost of shipping one unit from source i to destination j.
- The problem is to find a shipping programme that meets the required demands and minimises the total shipping cost.

- Initially, I'll assume that total supply equals total demand, i.e. that
$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$
- I'll show later that this requirement can easily be dropped.
- The transportation problem is a special case of a **network flow problem** where the nodes of the network are the shipping routes from each supply point to each demand point.
- Here's an example I'll refer to shortly: supppose I have 3 supply points and 3 demand points where each supply point has 20 units of the product and the demands are 10,30 & 20 units.
- The network associated with this problem is shown in Fig. 5.

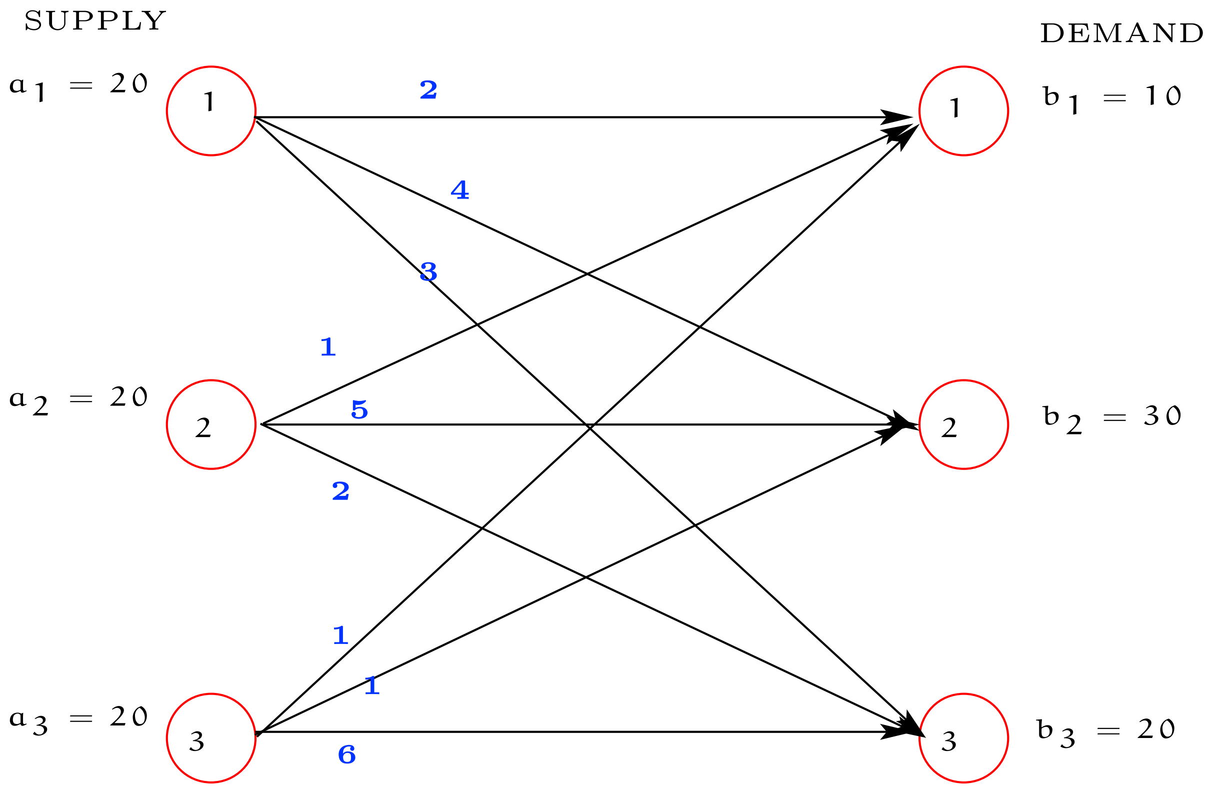

Figure 5: Transportation network with three supply & three demand points. The unit shipping costs are given in **blue** on each arrow.

- If I define the decision variables to be x_{ij}, the amount shipped from source i to destination j then a transportation problem can be formulated as the following LP:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} \qquad \sum_{j=1}^{n} x_{ij} = a_i, \text{for } i = 1, \ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \text{for } j = 1, \ldots, n.$$

$$x_{ij} \geq 0, \text{for each } i \text{ and } j.$$

- The good news is that the transportation problem is already in standard form — so I could use the Simplex algorithm to solve it, i.e. pivot it to optimal form.
- The bad news is that even for moderate values for m & n, the LP can be very large.
- For example, if $m = 100$ and $n = 100$, there are 200 equality constraints and $10,000$ variables in the problem.

- To see the special LP structure of a transportation problem, consider the toy example in Fig. 5.
- The simplex tableau is:

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
0	2	4	3	1	5	2	1	1	6
20	1	1	1	0	0	0	0	0	0
20	0	0	0	1	1	1	0	0	0
20	0	0	0	0	0	0	1	1	1
10	1	0	0	1	0	0	1	0	0
30	0	1	0	0	1	0	0	1	0
20	0	0	1	0	0	1	0	0	1

(4.1)

- This tableau isn't in canonical form so I need to pivot to introduce columns of the identity matrix (with zeros at the top of those columns) and then use the Dual Simplex Method to make the numbers in the constant column non-negative.
- But **because of the special structure of the problem** I can bypass that (long) process.
- In fact, getting an initial basic feasible point for a transportation problem is very easy.
- To see how to do it, look again at the network diagram in Fig. 5.
- The following steps illustrate the general method, called the **NorthWest corner rule**, for finding a basic feasible point .

- Looking at Fig. 5, start with the first source node, S_1 (in the NW corner) and ship as much as possible to the first destination D_1.
- As S_1 has 20 units of supply and D_1 needs 10 units, I set $x_{11} = 10$.
- Now S_1 has 10 units left and I assign that to D_2 by setting $x_{12} = 10$.
- Next consider S_2 and ship as much as possible to D_2 by setting $x_{22} = 20$ which satisfies demand at D_2 and uses up all the supply at S_2.
- Now go to the last supply node, S_3 and ship all its supply to the last demand node, giving $x_{33} = 20$.

- If an assignment (other than the last one) simultaneously meets a demand and uses up a supply, I'll assign a shipment of 0 (zero) units from the **current** source to the **next** destination.
- In the present example, this means I set $x_{23} = 0$.
- It certainly can't do any harm & I'll explain the reason shortly.

- It isn't hard to see that the above procedure, the **NorthWest corner rule**, can be generalised to a general transportation problem with m supply and n demand points.
- I'll state it formally shortly.
- The feasible solution found with the NWC Rule will always have $m+n-1$ assignments (of values to some of the x_{ij}'s), some of which may be zero.
- Reason as follows;
  - ⋆ There are $m+n$ equality constraints;
    $\sum_{j=1}^{n} x_{ij} = a_i$, for $i = 1, \ldots, m$ and
    $\sum_{i=1}^{m} x_{ij} = b_j$, for $j = 1, \ldots, n$.
  - ⋆ But I require that total supply equals total demand, i.e. that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, so in fact there are only $m+n-1$ "linearly independent" equality constraints so there are $m+n-1$ basic variables (some of which may be zero).

- For my simple Example in Fig. 5, $m + n - 1 = 3 + 3 - 1 = 5$.
- I can show the feasible solution found with the NWC Rule in Fig. 6.

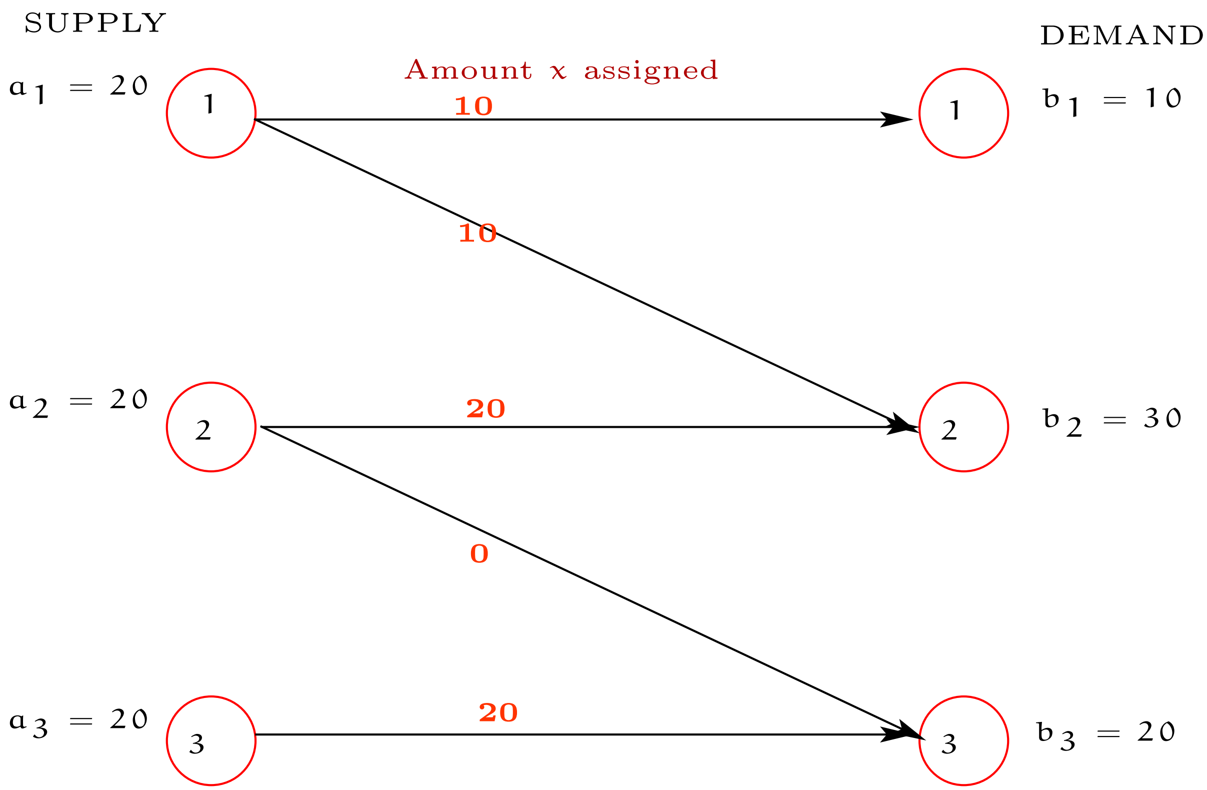

Figure 6: Sub-network showing feasible solution found with NWCR.

Exercise 4.1 *Pivot the starting tableau for this "toy" transportation problem to optimality using* `Pivot.m`*. How many pivots did you need?*

- I need to show how to pivot the starting tableau to get a basic feasible point that is the solution found by the NWCR.
- As I mentioned, the sum of the supply constraints equals the sum of the demand constraints, so one of the constraint equations is redundant.
- In fact the sum of the first three constraint rows equals the sum of the second three so I could for example add $-R_1 - R_2 - R_3 + R_4 + R_5$ to R_6 of the tableau to get a zero row.
- When I used the NWCR to construct the solution in Fig. 5, I constructed a feasible solution by sequentially assigning values to $m + n - 1$ variables so any assignment didn't affect the value of a variable already assigned.
- I can do the same sequence of assignments by pivoting on the starting simplex tableau.

- I'll use the same order (sequence) as in the NWCR & pivot to increase each chosen x_{ij} up to the level it has in the NWCR solution.

- Here's the starting tableau again.

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
0	2	4	3	1	5	2	1	1	6
20	1	1	1	0	0	0	0	0	0
20	0	0	0	1	1	1	0	0	0
20	0	0	0	0	0	0	1	1	1
10	1	0	0	1	0	0	1	0	0
30	0	1	0	0	1	0	0	1	0
20	0	0	1	0	0	1	0	0	1

- If I pivot with the circled element, I'll get $x_{11} = 10$ as needed.

- Giving the tableau :

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
-20	0	4	3	-1	5	2	-1	1	6
10	0	1	1	-1	0	0	-1	0	0
20	0	0	0	1	1	1	0	0	0
20	0	0	0	0	0	0	1	1	1
10	1	0	0	1	0	0	1	0	0
30	0	1	0	0	1	0	0	1	0
20	0	0	1	0	0	1	0	0	1

- If I pivot with the circled element, I'll get $x_{12} = 10$ as needed.

- Giving the tableau :

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
-60	0	0	-1	3	5	2	3	1	6
10	0	1	1	-1	0	0	-1	0	0
20	0	0	0	1	1	1	0	0	0
20	0	0	0	0	0	0	1	1	1
10	1	0	0	1	0	0	1	0	0
20	0	0	-1	1	1	0	1	1	0
20	0	0	1	0	0	1	0	0	1

- Pivoting with either of the circled elements makes $x_{22} = 20$ as needed — the **blue** pivot element will make x_{23} non-basic.
- But as I want x_{23} basic and zero at the next step, I'll pivot with Row 5 (use the **red** pivot element).

- Giving the tableau :

	$\mathbf{x_{11}}$	$\mathbf{x_{12}}$	$\mathbf{x_{13}}$	$\mathbf{x_{21}}$	$\mathbf{x_{22}}$	$\mathbf{x_{23}}$	$\mathbf{x_{31}}$	$\mathbf{x_{32}}$	$\mathbf{x_{33}}$
-160	0	0	4	-2	0	2	-2	-4	6
10	0	1	1	-1	0	0	-1	0	0
0	0	0	1	0	0	1	-1	-1	0
20	0	0	0	0	0	0	1	1	1
10	1	0	0	1	0	0	1	0	0
20	0	0	-1	1	1	0	1	1	0
20	0	0	1	0	0	1	0	0	1

Two more pivots to get $x_{23} = 0$ (and basic) and finally to get $x_{33} = 20$ gives the following two tableaux:

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
−160	0	0	2	−2	0	0	0	−2	6
10	0	1	1	−1	0	0	−1	0	0
0	0	0	1	0	0	1	−1	−1	0
20	0	0	0	0	0	0	1	1	1
10	1	0	0	1	0	0	1	0	0
20	0	0	−1	1	1	0	1	1	0
20	0	0	0	0	0	0	1	1	1

- and finally (either circled pivot will work as the two rows are the same, I've used the one in Row 3)

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
−280	0	0	2	−2	0	0	−6	−8	0
10	0	1	1	−1	0	0	−1	0	0
0	0	0	1	0	0	1	−1	−1	0
20	0	0	0	0	0	0	1	1	1
10	1	0	0	1	0	0	1	0	0
20	0	0	−1	1	1	0	1	1	0
0	0	0	0	0	0	0	0	0	0

- I can delete the all-zeros row to get a canonical form tableau whose associated basic feasible point is the same as that found using the NWCR earlier.

- For this example at least, this sequence of pivots shows that the feasible solution obtained by the NWCR is a basic feasible point.
- In fact, this will work for any NWCR solution as I can always do a sequence of pivots to get such a solution as a basic feasible point.
- Because a NWC Rule solution is obtained by sequentially assigning values to variables so that an assignment does not affect the value of a variable already assigned.
- So I can pivot to sequentially assign the same values to the variables.
- And I know that I must end up with a feasible solution, the NWCR solution.

4.1.1 The Transportation Tableau

- I could use a network diagram like Fig. 5 when using the NWC Rule to generate an inital feasible solution for a transportation problem.
- It is easier to use a special compact tabular format called a **transportation tableau.**
- Here's the transportation tableau for the current Example.

		b_1	b_2	b_3
		10	30	20
a_1	20	2^{10}	4^{10}	3
a_2	20	1	5^{20}	2^{0}
a_3	20	1	1	6^{20}

the (i, j) entry is $c_{ij}^{x_{ij}}$

- In a transportation tableau, I use **superscripts** on the cost coefficients to display the **values of the current basic variables**.
- The current nonbasic variables are of course zero so no need to display them as superscripts.
- For convenience I have listed the supplies and demands for the problem.
- Notice that
  - ⋆ the superscripts in each row add up to the supply for that row
  - ⋆ and the superscripts in each column add up to the demand for that column.

- Using the transportation tableau, here is the: **NWC Rule**.

  1. Pick the cost entry in the upper left (**NW**) corner & ship as much as possible by that route so that the supply is used up or the demand is met.

  2. ⋆ If the assignment just made uses up all the supply for that row, eliminate that row & return to Step 1.
     ⋆ If the assignment just made meets the demand for the column, eliminate that column & return to Step 1.
     ⋆ ∗ If the assignment just made uses up all the supply for that row i and meets the demand for that column j, assign $x_{i,j+1} = 0$ to the cost entry in $(i, j+1)$, unless the assignment just made was the final one.
       ∗ Eliminate row i and column j & return to Step 1.

- This is simple in practice. Here is the Example again. (I'll use **blue** to "strike out" a row or column.
- Assign as much as possible ($x_{11} = 10$) to the NW corner cost entry. This meets the demand at destination 1 so "strike out" column 1.

	10	30	20
20	$\mathbf{2^{10}}$	4	3
20	**1**	5	2
20	**1**	1	6

- The NW corner cost entry is now $c_{12} = 4$ so I assign as much as possible, namely $x_{12} = 10$ to this entry. This uses up the supply so "strike out" row 1.

	10	30	20
20	$\mathbf{2}^{\mathbf{10}}$	$\mathbf{4}^{\mathbf{10}}$	**3**
20	**1**	5	2
20	**1**	1	6

- Now I assign $x_{22} = 20$ to the remaining NW corner $c_{22} = 5$. This simultaneously uses up all the supply & meets the demand so I assign $x_{23} = 0$ and "strike out" both row 2 & column 2.

	10	30	20
20	$\mathbf{2}^{\mathbf{10}}$	$\mathbf{4}^{\mathbf{10}}$	$\mathbf{3}$
20	$\mathbf{1}$	$\mathbf{5}^{\mathbf{20}}$	$\mathbf{2}^{\mathbf{0}}$
20	$\mathbf{1}$	$\mathbf{1}$	6

- Only one cost entry remains so assign $x_{33} = 20$ as the final assignment.

	10	30	20
20	2^{10}	4^{10}	3
20	1	5^{20}	2^{0}
20	1	1	6^{20}

- Positions (i, j) in the transportation tableau are called **basic positions** if they correspond to a basic variable.
- So basic positions are those with an assigned superscript.
- Positions without a superscript are called **nonbasic positions**.

- The NWC Rule ignores the costs of the links so, while it produces an initial feasible point, it is often a high cost one.
- I'll look at other methods for finding intial feasible points shortly but first; given an intial feasible point, how do I solve the transportation problem?
- I could work with the Simplex tableau but that is typically very large and applying the Simplex algorithm directly to the tableau ignores the special structure of the transportation problem and is slow.
- Instead I'll introduce the **Transportation Algorithm** which exploits the special structure of the problem.
- It is based on (and as I will show, is derived from) LP Duality.

4.2 The Transportation Algorithm

The Transportation Algorithm is a nice (and useful) application of Duality. I'll put all the duality theory in a separate subsection.

You can skip the theory (though you shouldn't) and go directly to Slide 332.

4.2.1 Duality Applied to the Transportation Problem

I'll restate the Transportation Problem algebraically:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} \qquad \begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i, \text{for } i = 1, \dots, m. \\ \sum_{i=1}^{m} x_{ij} &= b_j, \text{for } j = 1, \dots, n. \\ x_{ij} &\geq 0, \text{for each } i \text{ and } j. \end{aligned}$$

The problem is already in standard form. To find the dual remember that I showed in Sec. 3.4.1 that the dual of a standard form LP

$$\begin{aligned} \min \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

is (3.1):

$$\begin{aligned} \max \quad & \mathbf{b}^{\mathsf{T}}\mathbf{y} \\ \text{subject to} \quad & A^{\mathsf{T}}\mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \quad \text{free.} \end{aligned} \tag{4.2}$$

I need to rewrite the Transportation Problem in matrix notation to apply this transformation. I'll re-display the tableau for the toy Transportation Problem above; Eq. 4.1:

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
0	2	4	3	1	5	2	1	1	6
20	1	1	1	0	0	0	0	0	0
20	0	0	0	1	1	1	0	0	0
20	0	0	0	0	0	0	1	1	1
10	1	0	0	1	0	0	1	0	0
30	0	1	0	0	1	0	0	1	0
20	0	0	1	0	0	1	0	0	1

(4.3)

The pattern is clear.

- The matrix A has $m+n=3+3=6$ rows and $m \times n = 3 \times 3 = 9$ columns.
- In the first $m=3$ rows, the string of $n=3$ ones $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ is repeated in each row (otherwise all zeros).
  - ⋆ The string of n ones moves $n=3$ columns to the right in each successive row.
- The remaining $n=3$ rows consist of the $n \times n$ identity matrix $I_n = I_3$ repeated $n=3$ times.

- So $A = \begin{bmatrix} 1 & 1 & 1 & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 1 & 1 & 1 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \vdots & 1 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 1 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix}$

  and $b^T = \begin{bmatrix} 20 & 20 & 20 & 10 & 30 & 20 \end{bmatrix}$.

As another example, if I had $m = 2$ and $n = 3$ then

$$A = \begin{bmatrix} 1 & 1 & 1 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$\mathbf{b}^T = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 & b_3 \end{bmatrix}$$

where (confusingly) I have used $\mathbf{b}$ as the constant column (LP notation) and b_1, b_2, b_3 for the demands in the Transportation Problem.

So in the general case, writing I_n as the $n \times n$ identity matrix, $e_n^T = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}$ (n ones) and $z_n^T = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}$ (n zeros) I can write A in "block matrix" notation as an $(m+n) \times mn$ matrix

$$A = \begin{bmatrix} e_n^T & z_n^T & z_n^T & \ldots & z_n^T & z_n^T \\ z_n^T & e_n^T & z_n^T & \ldots & z_n^T & z_n^T \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ z_n^T & z_n^T & z_n^T & \ldots & e_n^T & z_n^T \\ z_n^T & z_n^T & z_n^T & \ldots & z_n^T & e_n^T \\ I_n & I_n & I_n & \ldots & I_n & I_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The cost coefficients $\mathbf{c}$ are just the column vector of length mn whose transpose is $\mathbf{c}^T =$
$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} & c_{21} & c_{22} & \dots & c_{2n} & \dots & c_{m1} & c_{m2} & \dots c_{mn} \end{bmatrix}$$
So the Dual of the Transportation Problem is (using the rule (4.2))

$$\begin{aligned} \max \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & A^T\mathbf{y} \le \mathbf{c} \\ & \mathbf{y} \quad \text{free.} \end{aligned}$$

which, when I substitute in for A, $\mathbf{b}$ and $\mathbf{c}$ translates to (writing the free variable $\mathbf{y} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$ where $\mathbf{u}$ is of length m and $\mathbf{v}$ is of length n):

Maximise $\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$

(or just $\mathbf{a}^T \mathbf{u} + \mathbf{b}^T \mathbf{v}$)

where the vectors $\mathbf{u}$ and $\mathbf{v}$ are free (not required to be non-negative)

subject to (remember e_n and z_n are column vectors):

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$$\begin{bmatrix} e_n & z_n & \dots & z_n & z_n & I_n \\ z_n & e_n & \dots & z_n & z_n & I_n \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ z_n & z_n & \dots & e_n & z_n & I_n \\ z_n & z_n & \dots & z_n & e_n & I_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ \underline{\mathbf{v}} \end{bmatrix} \le \begin{bmatrix} \underline{\mathbf{c}}_1 \\ \underline{\mathbf{c}}_2 \\ \vdots \\ \underline{\mathbf{c}}_m \end{bmatrix}.$$

I've used a shorthand $\underline{\mathbf{c}}_1^T = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \end{bmatrix}$, $\underline{\mathbf{c}}_2^T = \begin{bmatrix} c_{21} & c_{22} & \dots & c_{2n} \end{bmatrix}, \dots, \underline{\mathbf{c}}_{\mathbf{i}}^{\mathbf{T}} = \begin{bmatrix} c_{i1} & c_{i2} & \dots & c_{in} \end{bmatrix}, \dots,$ $\underline{\mathbf{c}}_m^T = \begin{bmatrix} c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$.

Using block matrix multiplication, this translates into a set of vector inequalities:

$$
\begin{aligned}
u_1 \underline{\mathbf{e}}_n + \underline{\mathbf{v}} &\le \underline{\mathbf{c}}_1 \\
u_2 \underline{\mathbf{e}}_n + \underline{\mathbf{v}} &\le \underline{\mathbf{c}}_2 \\
\vdots &\le \vdots \\
\mathbf{u_i} \underline{\mathbf{e}}_\mathbf{n} + \underline{\mathbf{v}} &\le \underline{\mathbf{c}}_\mathbf{i} \\
\vdots &\le \vdots \\
u_m \underline{\mathbf{e}}_n + \underline{\mathbf{v}} &\le \underline{\mathbf{c}}_m
\end{aligned}
$$

and finally to

$$u_1 + v_1 \le c_{11}, u_1 + v_2 \le c_{12}, \dots, u_1 + v_n \le c_{1n},$$
$$u_2 + v_1 \le c_{21}, u_2 + v_2 \le c_{22}, \dots, u_2 + v_n \le c_{2n},$$
$$\vdots$$
$$\mathbf{u_i + v_1 \le c_{i1}, u_i + v_2 \le c_{i2}, \dots, u_i + v_n \le c_{in}.} \ ,$$
$$\vdots$$
$$u_m + v_1 \le c_{m1}, u_m + v_2 \le c_{m2}, \dots, u_m + v_n \le c_{mn}.$$

A much neater way to write this set of mn constraints is just

$$\mathbf{u_i + v_j \le c_{ij}, \quad i = 1, \dots, m, \quad j = 1, \dots, n.}$$

4.2.2 The Dual Transportation Problem

(You can jump back to the beginning of the duality algebra here: Slide 318.)

So the dual LP for the Transportation Problem is:

$$\max \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j = \mathbf{a}^T \mathbf{u} + \mathbf{b}^T \mathbf{v}$$

subject to

$$u_i + v_j \le c_{ij}, \quad i = 1, \dots, m, \quad j = 1, \dots, n$$

u&v free.

What use is all this algebra?

- Remember that the **optimal** objective function values for the primal & dual problems are equal (and only the optimal ones!).
- Write $p(\mathbf{x}) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ and $d(\mathbf{u}, \mathbf{v}) = a^T \mathbf{u} + b^T \mathbf{v}$.
- So one way to show that a feasible $\mathbf{x}$ is optimal is to show that there is a feasible dual vector $(\mathbf{u}, \mathbf{v})$ such that
$p(\mathbf{x}) - d(\mathbf{u}, \mathbf{v}) = 0$.
- Substituting,

$$p(\mathbf{x}) - d(\mathbf{u}, \mathbf{v}) \equiv \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} - \left(\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \right).$$

As $\mathbf{x}$ is primal feasible I have $a_i = \sum_{j=1}^{n} x_{ij}$ and $b_j = \sum_{i=1}^{m} x_{ij}$.

So, $p(\mathbf{x}) - d(\mathbf{u}, \mathbf{v}) = \displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} u_i - \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} v_j.$

- Simplifying;

$$p(\mathbf{x}) - d(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - u_i - v_j) x_{ij}. \tag{4.4}$$

- If $\mathbf{x}$ is a basic feasible solution for the Transportation Problem, suppose I find vectors $\mathbf{u}$ and $\mathbf{v}$ so that $c_{ij} - u_i - v_j = 0$ for each basic position (i, j).

- It follows immediately from (4.4) that $p(\mathbf{x}) = d(\mathbf{u}, \mathbf{v})$ as $x_{ij} = 0$ for the non-basic positions.

- So

  ⋆ provided that $\mathbf{u}$ and $\mathbf{v}$ are dual feasible (satisfy $u_i + v_j \leq c_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$),

  ⋆ and $c_{ij} - u_i - v_j = 0$ for each basic position (i, j)

  $\mathbf{x}$ is optimal for the the Transportation Problem.

4.2.3 Application of the Duality Result

- I've shown that the condition for a transportation tableau to be optimal is that:
  - ⋆ If I solve for a set of dual variables $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ satisfying the $m + n - 1$ equations:

$$c_{ij} = u_i + v_j \qquad \text{for each } \underline{\text{basic}} \text{ position.}$$

  - ⋆ the differences $c_{ij} - u_i - v_j \geq 0$ for <u>all</u> positions in the transportation tableau.
- In other words, calculate the $m + n$ dual variables $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ from the current transportation tableau (I'll explain how) then check if the differences $c_{ij} - u_i - v_j$ are non-negative for the nonbasic positions.
  - ⋆ The differences are zero for the basic positions by definition.

- If the initial transportation tableau is not optimal then a simple update rule is used to move to a better solution.
- I update the $m + n$ dual variables $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$.
- If the updated differences $c_{ij} - u_i - v_j$ are all non-negative I know that I have an optimal solution.
  - ⋆ This is because $c_{ij} - u_i - v_j = 0$ for all basic positions if and only if the primal objective equals the dual objective solution.
  - ⋆ If $c_{ij} - u_i - v_j \geq 0$ for all nonbasic positions then the solution is dual feasible.
  - ⋆ If both hold then by the Duality Theorem x is optimal.

4.2.4 Calculating the Dual Variables u's and v's

- I'll illustrate the procedure with the current Example.
- Here's the initial transportation tableau (found using the NWC Rule) again.

	10	30	20
20	2^{10}	4^{10}	3
20	1	5^{20}	2^{0}
20	1	1	6^{20}

- The basic positions are those with a superscript; $(1,1)$,$(1,2)$, $(2,2)$, $(2,3)$ and $(3,3)$.
- Remember that the superscript gives the assignment (x_{ij} value) for that link, the base number is the cost c_{ij} for the link.

- I need to find dual vectors **u** and **v** satisfying

$$c_{ij} = u_i + v_j \quad \text{for each basic position } (i, j).$$

- So for the example, **u** and **v** must satisfy the system of equations;

$$\begin{aligned}
u_1 + v_1 &= 2 & &= c_{11}\\
u_1 + v_2 &= 4 & &= c_{12}\\
u_2 + v_2 &= 5 & &= c_{22}\\
u_2 + v_3 &= 2 & &= c_{23}\\
u_3 + v_3 &= 6 & &= c_{33}
\end{aligned}$$

- This looks like a lot of work — but it isn't!

- As there is one less (5) equation than the (6) unknowns, I can choose **any** value (say zero) for u_1 and just successively substitute into the 5 equations to read off the values of the other u_i's and v_j's.
- So
  - ⋆ $u_1 = 0$ gives $v_1 = 2$ & $v_2 = 4$.
  - ⋆ $v_2 = 4$ gives $u_2 = 1$.
  - ⋆ $u_2 = 1$ gives $v_3 = 1$.
  - ⋆ $v_2 = 1$ gives $u_3 = 5$.
- In fact, I don't even need to write down the set of equations for the u_i's and v_j's.
- I can work directly with the transportation tableau.

- Just set $u_1 = 0$ in the transportation tableau below and follow the succession of substitutions:

2^{10}	4^{10}	3	**0**	u_1
1	5^{20}	2^{0}	**1**	u_2
1	1	6^{20}	**5**	u_3
2	**4**	**1**		
v_1	v_2	v_3		

4.2.5 Calculating the Differences $c_{ij} - u_i - v_j$

- This is straightforward
  - ⋆ The differences are automatically zero for the basic positions.
  - ⋆ Just subtract the u_i & v_j value for each nonbasic row and column from the cost c_{ij} at that position.
  - ⋆ In fact, I'll overwrite the costs in the transportation tableau with these differences – calling them **adjusted cost entries**.
  - ⋆ I'll show on the next Slide that working with the adjusted costs gives the same solution as working with the original ones.
  - ⋆ **Just reverse the argument that led to (4.4).**
  - ⋆ And I can use the adjusted costs to calculate the change in the cost for a new solution.

- This is very convenient.

Check that I can use the adjusted costs

- The primal objective using the adjusted costs, ($\hat{p}(\mathbf{x})$ say) is:

$$\begin{aligned}
\hat{p}(\mathbf{x}) &= \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij} - u_i - v_j)x_{ij} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n} u_i x_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n} v_j x_{ij} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{m} u_i \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} v_j \sum_{i=1}^{m} x_{ij} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{m} u_i a_i - \sum_{j=1}^{n} v_j b_j \\
&= p(\mathbf{x}) - \mathbf{a}^T u - \mathbf{b}^T v = p(\mathbf{x}) + \text{constant}.
\end{aligned}$$

- So minimising $\hat{p}(\mathbf{x})$ w.r.t. x over the feasible set is equivalent to minimising $p(\mathbf{x})$ w.r.t. x over the feasible set.

- So the updated transportation tableau for the Example is:

$$
\begin{array}{|ccc|cc}
\hline
0^{10} & 0^{10} & 2 & 0 & u_1 \\
\mathbf{-2} & 0^{20} & 0^{0} & 1 & u_2 \\
\mathbf{-6} & \mathbf{-8} & 0^{20} & 5 & u_3 \\
\hline
2 & 4 & 1 & & \\
v_1 & v_2 & v_3 & &
\end{array}
\tag{4.5}
$$

- So the solution is infeasible as three of the nonbasic entries have negative updated costs.
- I'll need to apply an update step to the transportation tableau.
- How?
- I'll explain in the next subsection.

4.2.6 Updating the Transportation Tableau.

- I need to change the flow in the network so as to reduce the total cost.
- Remember I can work with the adjusted costs (the "differences").
- So I need to (using LP terminology) increase a nonbasic variable from zero.
- Or equivalently, select a link in the network.
- What link?
- Just pick the nonbasic position in the transportation tableau with the most negative adjusted cost.
- For the current Example, the $(3, 2)$ position whose cost is -8.

- So I will increase the flow on the $(3, 2)$ link from zero — introducing a new route.
- I'll indicate this on the network diagram with a dotted line.

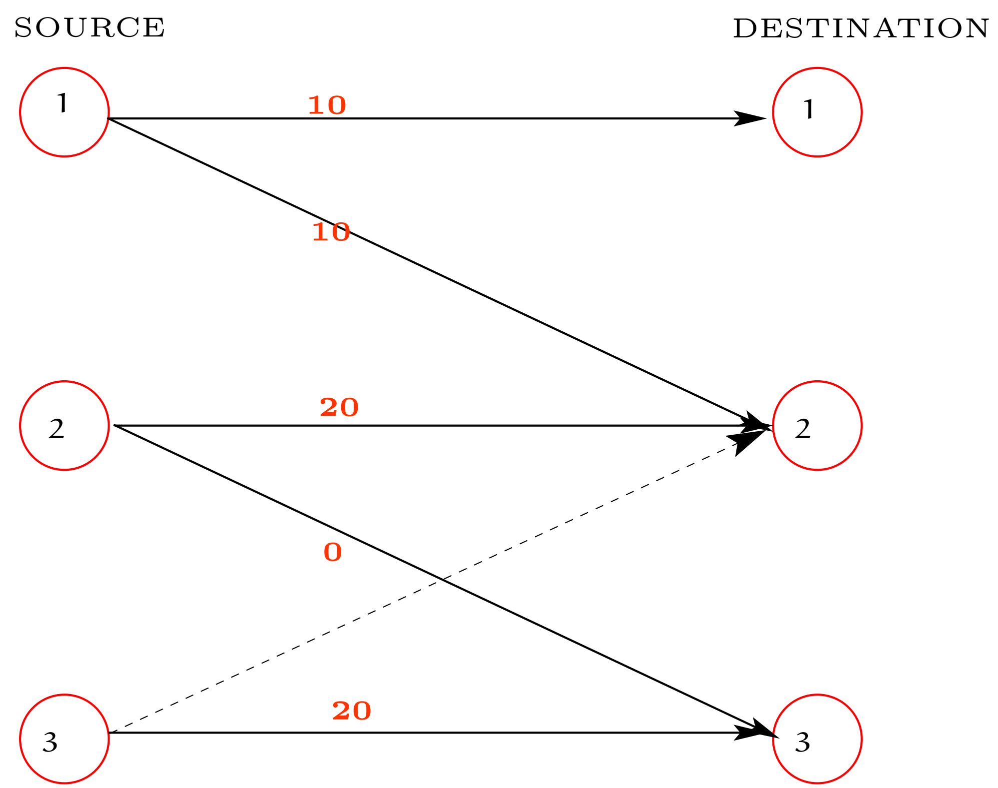

Figure 7: Introducing a New Shipping Route.

- I need to adjust the other flows to keep the solution feasible.
- This is easy using the network diagram as adding the extra (dotted) arrow creates a **loop** in the network.
  - ⋆ Starting at one node I can follow a path of successive links
  - ⋆ (sometimes going from Right to Left)
  - ⋆ and return to the start node.
- The loop in Fig. 7 is $S_3 \to D_2 \to S_2 \to D_3 \to S_3$.
- When I add a link to a subnetwork with $n + m - 1$ links (and no loops) I always create a loop as I now have a source node with an extra leaving link and a destination node with an extra arriving link.
- The NWC Rule will never have a loop in its set of basic positions.

- The key insight:
  - ⋆ If I start with the dotted (extra) arc & alternately add and subtract a positive flow t to the links in the loop
  - ⋆ Then the new solution is feasible provided all the flows are still non-negative.
- In the Example, suppose I increase the flow on the dotted link from 0 to t, i.e. $x_{32} \to t$.
- Then, following the loop, I have $x_{22} \to 20 - t$, $x_{23} \to 0 + t$, $x_{33} \to 20 - t$.
- All the other links in the current solution are unchanged as they are "out of the loop".

- The new solution is guaranteed to satisfy the supply & demand constraints.
- So it is feasible, provided that t satisfies $20 - t \geq 0$.
- So the new solution is feasible for $t \leq 20$.
- Going round the loop in the Figure, if I take $t = 20$, I get the new feasible solution.
- I can compute the change in the total cost from the transportation tableau Eq. 4.5.
- The only non-zero adjusted cost in the loop is $c_{32} = -8$ and x_{32} has increased to 20.
- So $\Delta\text{Cost} = 20 \cdot (-8) = -160$.
- My new solution has a cost that is 160 less than the starting solution.

- Two basic variables (x_{22} and x_{33}) are now zero — remember that zero basic variables are called "degenerate".
- I need to demote one of them to nonbasic status (the number of basic variables is always $m + n - 1$ (5 for the Example) — I arbitrarily choose to keep x_{33} basic and remove x_{22} from the basis.

- In terms of the transportation tableau, I just start at the position with the most negative adjusted cost and create a loop connecting **basic** positions only.
  - ⋆ I've highlighted the "loop" positions in **blue** starting with $(3, 2)$ in **red**:

0^{10}	0^{10}	2
-2	$\mathbf{0^{20}}$	$\mathbf{0^{0}}$
-6	$\mathbf{-8}$	$\mathbf{0^{20}}$

- What are the "rules" for forming a loop? Very simple;
  - ⋆ **Links connect basic positions only.**
  - ⋆ **Links in the transportation tableau must be horizontal or vertical, not diagonal.**
  - ⋆ **Links in the transportation tableau are alternately horizontal and vertical.**

- These simple rules make it easy to find the loop starting at the selected position.
- Here's a randomly generated example with $m = 5$ and $n = 7$ (so 11 basic positions). I'll indicate the 11 randomly located basic positions with $*$'s and the randomly chosen non-basic start position with $\circ$.

*	.	.	*	.	.	.
.	*	*	.	*	.	.
*	.	.	.	○	.	.
.	.	*	.	.	.	*
.	*	.	*	.	*	.

- It can be shown that there is only one loop possible from any start position.

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- For a large transportation tableau, "finding the loop" isn't easy.
- But for the small problems in this course it is..
- Here are a few more.

*	.	.	*	.	.	*
.	*	.	.	○	.	.
*	.	.	.	.	*	.
.	.	*	.	.	.	*
.	*	.	*	*	.	.

*	.	.	*	.	.	.
.	*	*	.	.	.	.
*	.	.	.	.	*	.
.	○	*	.	.	.	*
.	*	.	*	.	*	.

- Easy...

- The new subnetwork diagram for the loop example problem is:

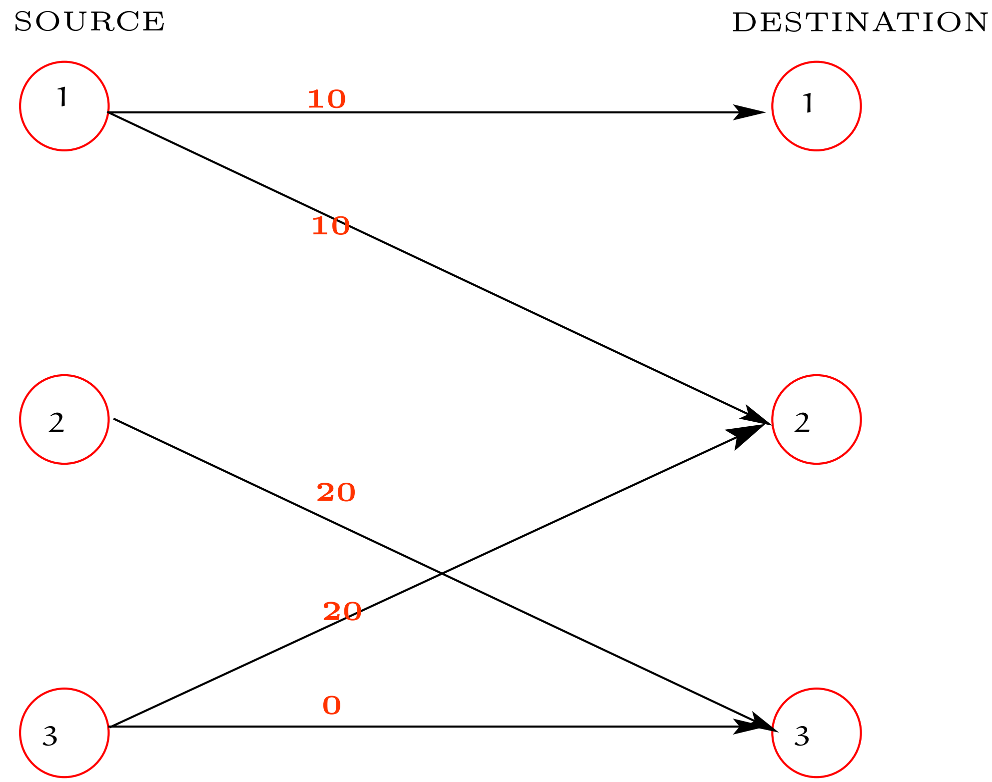

Figure 8: Subnetwork for the new solution.

- The transportation tableau for the new solution is

0^{10}	0^{10}	2
-2	0	0^{20}
-6	-8^{20}	0^{0}

- Suppose I had chosen the cost entry equal to -6 (at position $(3,1)$ in the starting transportation tableau.
- Then the loop would have been $S_3 \to D_1 \to S_1 \to D_2 \to S_2 \to D_3 \to S_3$.
- Equivalent to the succession of positions $(3,1)$,$(1,1)$, $(1,2)$,$(2,2)$,$(2,3)$,$(3,3)$,$(3,1)$.
- I've highlighted the "loop" positions for this choice in **blue** starting with $(3,1)$ in **red**:

$\mathbf{0^{10}}$	$\mathbf{0^{10}}$	2
-2	$\mathbf{0^{20}}$	$\mathbf{0^{0}}$
$\mathbf{-6}$	-8	$\mathbf{0^{20}}$

- The max amount that I can shift around this loop is $t = 10$ and the resulting decrease in the cost is $10 \cdot (-6) = -60$.

- After this alternative (and inferior) update choice, the transportation tableau would have updated to:

$$\begin{array}{|ccc|}\hline 0^{0} & 0^{20} & 2 \\ -2 & 0^{10} & 0^{10} \\ -6^{10} & -8 & 0^{10} \\ \hline\end{array}$$

- N.B. the positions with cost entries of -2 and -8 are not in the loop so I didn't change the associated x_{ij}'s.
- Exercise: Repeat the calculation starting at the cost entry equal to -2 (at position $(2,1)$) in the starting transportation tableau.

4.2.7 Getting an Optimal Transportation tableau

- Am I finished?
- To find out I need to recalculated the adjusted costs for the new transportation tableau and check if they are all nonnegative.
- Here's the transportation tableau for the new solution again:

$$\boxed{\begin{array}{ccc} 0^{10} & 0^{10} & 2 \\ -2 & 0 & 0^{20} \\ -6 & -8^{20} & 0^{0} \end{array}}$$

- Updating the values of u and v by setting $u_1 = 0$ and "cascading" through the transportation tableau gives:

$$\begin{array}{|ccc|cc} \hline 0^{10} & 0^{10} & 2 & 0 & u_1 \\ -2 & 0 & 0^{20} & -8 & u_2 \\ -6 & -8^{20} & 0^{0} & -8 & u_3 \\ \hline 0 & 0 & 8 & & \\ v_1 & v_2 & v_3 & & \end{array} \tag{4.6}$$

- Now calculate new adjusted cost coefficients using the formula $c_{ij} - u_i - v_j$ as before (remember the adjusted cost coefficients for the basic positions are automatically zero) gives the new transportation tableau :

0^{10}	$\mathbf{0^{10}}$	$\mathbf{-6}$
6	8	$\mathbf{0^{20}}$
2	$\mathbf{0^{20}}$	$\mathbf{0^{0}}$

- Only one dual constraint is violated (negative adjusted cost entry) so I look for the unique loop starting at $(1, 3)$.
- I've highlighted the "loop" positions in **blue** starting with $(3, 1)$ in **red**.
- **I had to skip the $(2, 3)$ position as there is no link from supply 2 to any demand other than Demand 3.**
- In the transportation tableau, only one basic position in row 2.

- Because the $(3, 3)$ position has $x_{33} = 0$, $t = 0$ is the maximum amount that can be alternately added to & subtracted from the basic positions in the loop.
- Performing this "degenerate" shift around the loop just causes x_{13} to become basic and x_{33} to become nonbasic.
- So the new transportation tableau is:

0^{10}	0^{10}	-6^{0}	0	u_1
6	8	0^{20}	6	u_2
2	0^{20}	0	0	u_3
0	0	-6		
v_1	v_2	v_3		

- The new u_i's and v_j's are calculated and shown in the margins.

- To see if this new solution is optimal, I calculate new adjusted costs using the formula $c_{ij} - u_i - v_j$ as before (remember the adjusted cost coefficients for the basic positions are automatically zero) gives the new transportation tableau :

$$\boxed{\begin{array}{ccc} 0^{10} & 0^{10} & 0^{0} \\ 0 & 2 & 0^{20} \\ 2 & 0^{20} & 6 \end{array}}$$

- I have an optimal solution!

4.2.8 Formal Statement of the Transportation Algorithm

- It only remains to formally state the Transportation Algorithm.

1. Find an initial basic feasible point $\mathbf{x}$.
2. ⋆ For the current solution $\mathbf{x}$ and the current cost coefficients c_{ij}, find a dual vector $(\mathbf{u}, \mathbf{v})$ such that $u_i + v_j = c_{ij}$ for all basic positions (i, j).
   ⋆ Calculate the adjusted cost coefficients (a.c.c.) for all positions (i, j).
3. ⋆ If each a.c.c is ≥ 0, STOP.
   ⋆ ELSE Pick the position with the most negative a.c.c and find the unique loop starting there (with all other positions basic).
4. ⋆ Shift as much as possible around the loop to get a new basic feasible point $\mathbf{x}$.
   ⋆ GOTO Step 2 with the a.c.c's as the new costs.

4.2.9 Other Methods for Finding an Initial Feasible Point

- In addition to the NWC Rule , two other methods are often used, the Smallest Cost Entry Method and Vogel's Method.
- I'll only describe the first.

The Smallest Cost Entry Method Again assuming that supply equals demand:

1. Pick the smallest remaining cost entry in the transportation tableau and ship as much as possible by that route so that the supply is used up or the demand met.
2. - If the assignment just made uses up the supply for the row, eliminate that row and GOTO Step 1.
   - If the assignment just made meets the demand for that column, eliminate that column and GOTO Step 1.
   - If the assignment just made uses up the supply for the row **and** meets the demand for that column and it is not the last assignment
     - ⋆ Assign $x_{ij} = 0$ to the next smallest cost entry in the column.
     - ⋆ Eliminate the row **and** the column and GOTO Step 1.

I'll illustrate the SCEM with an example:

	15	10	10	5	30
10	3	6	8	11	5
15	1	9	3	2	7
30	4	2	8	25	15
5	9	1	4	9	8
10	2	4	2	11	1

See `http://jkcray.maths.ul.ie/ms4303/Scans/TransportationExample.pdf` for a solution.

4.2.10 Unequal Supply & Demand

- This is of course much more likely than exact matching of supply & demand.
- Fortunately it is easy to handle.
- If supply exceeds demand just create a fictitious demand equal to the surplus supply and assign zero cost to shipping to that new demand..
- This adds an extra **column** to the transportation tableau with zeros in the new column.
- Now apply the Transportation Agorithm.
- Finally, delete the fictitious column once the problem is solved.

- If supply is less than demand, just add a fictitious supply equal to the shortall, with zero cost.
- This adds an extra **row** to the transportation tableau with zeros in the new row.
- Again, apply the Transportation Agorithm.
- And, once the augmented problem is solved, delete the fictitious row.
- Done.

4.2.11 Forbidden Links

- It might happen that certain sources may not (for geographical reasons perhaps) ship to certain destinations.
- How to enforce these restrictions?
- One simple solution, assign a cost of $+\infty$ to these positions in the transportation tableau.
- In practice it is more convenient to assign a cost of N where N is not given a value but just taken to be so large that N minus any value is still much bigger than any other value not involving N.
- It is intuitively obvious that the transportation algorithm will never make these positions basic (assign positive flow to these links).

4.3 Exercises for Ch.4

1. - A company has 3 warehouses that are usd to supply 3 shops with a product.
   - Each warehouse has 20 units of the product and shops 1,2 & 3 need 20, 15 and 15 units respectively.
   - The per-unit costs of shipping from warehouse 1 to shops 1 & 2 are 5 & 7, respectively.
   - Warehouse 1 cannot ship to shop 3.
   - The per-unit costs of shipping from warehouse 2 to to shops 1, 2 & 3 are 6, 2 and 9, respectively.
   - The per-unit costs of shipping from warehouse 3 to to shops 1, 2 & 3 are 1,1 & 5, respectively.

(a) Write down a transportation tableau with supplies, demands & costs specified so that an optimal solution will provide the lowest cost shipping schedule.

(b) Find an optimal shipping schedule.

http://jkcray.maths.ul.ie/ms4303/Scans/Exercise5.1.pdf

2. Use the transportation algorithm to solve the transportation problems for the following transportation tableaux with supplies and demands as shown. (Check that total supply equals total demand and where it doesn't use fictitious supply or demand as appropriate.)

	20	5	5	10
10	9	3	4	5
15	2	5	1	2
15	3	9	1	5

	10	15	10	15
20	7	3	8	5
20	4	1	6	2

	20	15	20
20	7	1	2
20	1	5	5
25	4	1	4

3. Consider the following transportation tableau with an initial

basic feasible point as shown by the superscripts:

1^5	1^5	1
2^{10}	4	3^5
4	4^{10}	1

(a) Stating with the displayed basic feasible point , apply the algorithm explained in the Notes to get an optimal transportation tableau.

(b) Write down the optimal vector and the minimum cost.

(c) Suppose that it isn't possible to use the shipping route from source 1 to destination 2. Find an optimal solution subject to this restriction.

5 Sensitivity Analysis in LP

- In this Chapter I'll explain how to track the changes in the optimal solution due to changes in the data or the addition of extra constraints.
- The term used is "sensitivity analysis".
- The good news is that sensitivity analysis doesn't require re-solving the changed LP.
- Often the optimal tableau for one problem can be used to find an optimal solution for the "perturbed" problem.

5.1 The Dear Beer Co. Problem

- I'll use the Dear Beer Co. problem (Example 1.3) to illustrate sensitivity analysis techniques.
- You have already seen on Slide 81 the following algebraic description:

$$\begin{aligned} \max 6x_1 + 5x_2 + 3x_3 + 7x_4 \\ \text{subject to} \\ 1x_1 + 1x_2 + 0x_3 + 3x_4 &\leq 50 \\ 2x_1 + 1x_2 + 2x_3 + 1x_4 &\leq 150 \\ 1x_1 + 1x_2 + 1x_3 + 4x_4 &\leq 80 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(The variables x_1, x_2, x_3, x_4 stand for the amount of each type of beer to be produced.)

- You have also seen that after adding slack variables s_1, s_2, s_3 (I labelled them x_5, x_6, x_7 earlier) I get the canonical form tableau :

$$P = \begin{array}{|r|ccccccc|}
\hline
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 \\
\hline
0 & -6 & -5 & -3 & -7 & 0 & 0 & 0 \\
\hline
50 & 1 & 1 & 0 & 3 & 1 & 0 & 0 \\
150 & 2 & 1 & 2 & 1 & 0 & 1 & 0 \\
80 & 1 & 1 & 1 & 4 & 0 & 0 & 1 \\
\hline
\end{array}$$

- Pivoting with the Simplex algorithm gives the optimal tableau :

$$P^* = \begin{array}{|c|ccccccc|} \hline & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 \\ \hline 380 & 0 & 0 & 0 & 7 & 3 & 1 & 1 \\ \hline 40 & 1 & 0 & 0 & -4 & 1 & 1 & -2 \\ 30 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \\ 10 & 0 & 1 & 0 & 7 & 0 & -1 & 2 \\ \hline \end{array}$$

- I showed on Slide 224 that once I know P and P^* it is easy to write down a "pivot matrix" Q such that $P^* = QP$.

- Check that $\mathsf{Q} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$.
- The first column of a pivot matrix is always the first column of the identity matrix.
- Columns 2,3, & 4 are just the columns of P^* corresponding to the identity columns of P (the slack variable columns in this case).
- The pivot matrix is a key tool in sensitivity analysis as I will show.

- In the following Sections I'll show how to analyse the effect on the optimal vector of:
  - ⋆ changes in production requirements,
  - ⋆ changes in available resources,
  - ⋆ changes in the selling price of a product.

5.2 Changes in Production Requirements

5.2.1 Changes in Nonbasic Variables

- When I discussed shadow prices in Section 3.3 I showed how an optimal vector changes because of changes in slack variables that are nonbasic in the optimal tableau.
- **The method I used then can be used with any nonbasic variable in the optimal tableau.**
- For example, in the Dear Beer Co. problem (on Slide 377), suppose I want one unit of Stout (x_4) to be produced.
- So instead of having the nonbasic variable $x_4 = 0$ in the optimal solution, I require that $x_4 = 1$.
- I want to move (with the minimum possible decrease in the objective value — remember this is a max problem) from the optimal point to another feasible point with $x_4 = 1$.

- Combining the x_4 column with the constant column gives the following tableau :

	x_1	x_2	x_3	s_1	s_2	s_3
380-7 x_4	0	0	0	3	1	1
40+4x_4	1	0	0	1	1	-2
30-x_4	0	0	1	-1	0	1
10-7x_4	0	1	0	0	-1	2

- When $x_4 = 1$ the constant column in this tableau stays nonnegative.
- So the tableau is still optimal.
- The constant column stays nonnegative as long as x_4 is less than the minimum row ratio for the x_4 column in P^*.
- Why?

- The new optimal vector is

$$\begin{bmatrix} 40 + 4x_4 \\ 10 - 7x_4 \\ 30 - x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 44 \\ 3 \\ 29 \\ 1 \end{bmatrix} \text{ when } x_4 = 1.$$

- The income has been **reduced from 380 to 373**,
  - ⋆ not surprising as, by insisting that $x_4 = 1$, I have selected a sub-optimal vector x.
  - ⋆ given that the problem is a max problem, sub-optimal choices result in lower values for the objective function.

- The example shows that the process can be summarised as:
  - ⋆ Given an optimal tableau P^*;
    - ∗ **If** the required increase in a nonbasic variable x_k is less than the minimum row ratio for the x_k column
    - ∗ **Then** to get a new optimal vector, set x_k to the required new value and keep all other nonbasic variables zero.
    - ∗ **Substitute** the new value of x_k into the basic variables to update them.

- Another example of the procedure; suppose that I need $s_2 = 5$ in the optimal solution (so 5 units of hops will be left unused).
- In the s_2 column in P^* I can see that the minimum row ratio is 40.
- So I write the new optimal vector as:

$$\begin{bmatrix} 40 - 1s_2 \\ 10 + 1s_2 \\ 30 \\ 0 \end{bmatrix} = \begin{bmatrix} 35 \\ 15 \\ 30 \\ 0 \end{bmatrix} \text{ when } s_2 = 5.$$

- And the income has been reduced from 380 to $380 - s_2 = 375$.

5.2.2 Increasing Basic Variables

- The analysis above shows how to deal with changes in a nonbasic variable provided the change is not bigger than the minimum row ratio.
- If the change in a nonbasic variable exceeds the minimum row ratio for that variable, some basic variable will become non-basic (the basis will change).
- To handle the case where a change in a nonbasic variable exceeds the minimum row ratio, I first need to look at how changes in **basic** variables in an optimal tableau affect the optimal solution.
- For the Dear Beer Co. problem suppose that I need the optimal amount of the **basic** x_1 to be increasd by 1 to 41.

- Here's the optimal tableau again:

$P^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
380	0	0	0	7	3	1	1
40	1	0	0	-4	1	1	-2
30	0	0	1	1	-1	0	1
10	0	1	0	7	0	-1	2

- It should be obvious that I can only increase a basic variable in an optimal tableau if there is at least one **negative** element in the row of P^* containing the "1" for the variable.
- The row containing the "1" for the variable (Row 1) corresponds to the equation: $40 = x_1 - 4x_4 + s_1 + s_2 - 2s_3$.
- Or equivalently $x_1 = 40 + 4x_4 - s_1 - s_2 + 2s_3$.

- To increase x_1 from its optimal value of 40, I need to increase either x_4 or s_3 from zero.
- The equation shows that I can increase x_1 up to 41 by
  - ⋆ either increasing x_4 from 0 to $\frac{1}{4}$
  - ⋆ or increasing s_3 from 0 to $\frac{1}{2}$.
- The best (optimal) change is the one that increases the objective function the least (corresponding to decreasing the optimal income the least as this is a max problem) .
- From P*, the objective function is $z = -380 + 7x_4 + 3s_1 + s_2 + s_3$, so the increase in x_4 would increase z by 7/4 while the increase in s_3 would increase z by only $\frac{1}{2}$.

- So I get a new optimal vector by letting $s_3 = \frac{1}{2}$ in P^* and keeping all other nonbasic variables zero.

- The new optimal vector is

$$\begin{bmatrix} 40 + 2s_3 \\ 10 - 2s_3 \\ 30 - s_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 41 \\ 9 \\ 29\frac{1}{2} \\ 0 \end{bmatrix} \text{ when } s_3 = \frac{1}{2}.$$

- The income has been **reduced from 380 to $379\frac{1}{2}$**,
  - ⋆ again not surprising as, by insisting that $x_1 = 41$, I have selected a sub-optimal vector x.
  - ⋆ again, given that the problem is a max problem, sub-optimal choices result in lower values for the objective function.

- I can summarise:
  - ⋆ An **increase** in a basic variable in row r of P^* is possible **only if some element (the k^{th}, (say)) of row r is negative; $a_{rk} < 0$.**
  - ⋆ The new optimal vector is found by increasing the nonbasic variable x_k, **where k is chosen to be the column with $a_{rk} < 0$ that maximises $\frac{c_k}{a_{rk}}$.**
  - ⋆ Provided that **the increase in x_k necessary to achieve the increase in the basic variable doesn't exceed the minimum row ratio for the x_k column.**
- In the example, s_3 is selected by this rule (compare $\frac{7}{-4}$ with $\frac{1}{-2}$).
- When working on a problem it is usually easier to reason your way to the conclusion (as I did on the preceding Slides) rather than mechanically apply the above rule.

- Another illustration; suppose I want the basic variable x_3 to increase to 35 from the optimal value of 30 found in P^*.
- In this case there is only one nonbasic variable that can be increased to give the required change, namely s_1 .
- Letting $s_1 = 5$, less than the minimum row ratio (40) for the s_1 column, I have the new optimal vector:

$$\begin{bmatrix} 40 - s_1 \\ 10 \\ 30 + s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 35 \\ 10 \\ 35 \\ 0 \end{bmatrix} \text{ when } s_1 = 5.$$

- Three equivalent questions:
  - ⋆ What is the new value of z?
  - ⋆ What is the new income value?
  - ⋆ How much has it cost the company to increase x_3 from the optimal value of 30 to the new value of 35?

5.2.3 Decreasing Basic Variables

- I can analyze decreases in variables that are basic in the optimal tableau using a similar procedure.
- For example, suppose that I need $x_2 = 8$ rather than the optimal value of 10.
- If I want to keep the row defining x_2 feasible, I need at least one entry in the row that is **positive** so that there is an increase to balance the decrease in x_2 .
- In this case the objective function z is $z = -380 + 7x_4 + 3s_1 + s_2 + s_3$ and the constraint row defining x_2 is $10 = x_2 + 7x_4 - s_2 + 2s_3$.
- So I can decrease x_2 from 10 to 8 by either increasing x_4 to 2/7 or by increasing s_3 to 1.

- Keeping the other nonbasic variables at zero and

$$\text{setting } x_4 = 2/7 \Rightarrow z = -380 + 7 \cdot 2/7 = -378$$

and

$$\text{setting } s_3 = 1 \Rightarrow z = -380 + 1 \cdot 1 = -379.$$

- The best choice is the latter, increase s_3 to 1. **Why?**
- The new optimal vector (from the optimal tableau P^*) is:

$$\begin{bmatrix} 40 + 2s_3 \\ 10 - 2s_3 \\ 30 - s_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \\ 29 \\ 0 \end{bmatrix} \quad \text{when } s_3 = 1.$$

- In summary:
  - ⋆ A **decrease** in a basic variable in row r of P^* is possible **only if some element (the k^{th}, (say)) of row r is positive; $a_{rk} > 0$.**
  - ⋆ The new optimal vector is found by increasing the nonbasic variable x_k, **where k is chosen to be the column with $a_{rk} > 0$ that minimises $\frac{c_k}{a_{rk}}$.**
  - ⋆ Provided that **the increase in x_k necessary to achieve the increase in the basic variable doesn't exceed the minimum row ratio for the x_k column.**
  - ⋆ Again, when working on a problem it is usually easier to reason your way to the conclusion (as I did on the preceding Slides) rather than mechanically apply the above rule.

    STOP **Stopped here** 10:00, Thursday Week 9

5.2.4 When a Nonbasic Variable becomes Basic and Exceeds its Minimum Row Ratio

- I'll finish the Section by combining the technique for changing nonbasic variables with the technique for changing basic variables to show how the optimal vector changes when the change needed in a nonbasic variable x_k is bigger than the minimum row ratio for the x_k column.
- Here's P^* again:

$P^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
380	0	0	0	7	3	1	1
40	1	0	0	-4	1	1	-2
30	0	0	1	1	-1	0	1
10	0	1	0	7	0	-1	2

- Suppose that I want to find the new optimal vector if $x_4 = 5$.
- This value for x_4 exceeds the minimum row ratio $(10/7)$ for the x_4 column.
- **I can't simply set $x_4 = 5$ in P^* and combine the x_4 column with the constant column as this would result in some negative constant column entries.**
- **Instead, I have to pivot in P^* to get x_4 up to the minimum row ratio of $10/7$ and make x_4 basic.**
- Once x_4 is basic I can analyse further increases in x_4 using the technique for increasing basic variables.

- When I pivot in P^* on the min. ratio row (row 3) & the x_4 column I find:

$$P' = \begin{array}{r|rrrrrrr} & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 \\ \hline 370 & 0 & -1 & 0 & 0 & 3 & 2 & -1 \\ \hline 320/7 & 1 & 4/7 & 0 & 0 & 1 & 3/7 & -6/7 \\ 200/7 & 0 & -1/7 & 1 & 0 & -1 & 1/7 & 5/7 \\ 10/7 & 0 & 1/7 & 0 & 1 & 0 & -1/7 & 2/7 \end{array}$$

- The variable x_4 is now basic.
- I can increase it from $10/7$ as the row in P' defining x_4 has at least one negative entry (in the s_2 column).
- If I increase s_2 up to 25 (less than the minimum row ratio of 320/3 for the s_2 column), I have x_4 =5 as asked for.

- So the new optimal vector is:

$$\begin{bmatrix} 320/7 - (3/7)s_2 \\ 0 \\ 200/7 - (1/7)s_2 \\ 10/7 + (1/7)s_2 \end{bmatrix} = \begin{bmatrix} 35 \\ 0 \\ 25 \\ 5 \end{bmatrix} \quad \text{when } s_2 = 25.$$

  The new z–value is $370 - 2 \times 25 = 320$.

- To see how efficient the technique is try adding the constraint $x_4 \geq 5$ to P^*.

- You'll find that it takes **two** iterations of DSM to find the optimal solution starting from the optimal tableau for the original problem and adding the constraint $x_4 \geq 5$.

- In summary:
  - ⋆ A **increase** in a nonbasic variable x_k whose minimum row ratio is in row r of P^* **above** that minimum row ratio can be accomplished **in two steps** by:
    - ∗ First pivoting on the (r, k) element of the tableau to make x_k basic.
    - ∗ Then using the technique above for increasing basic variables.

5.3 Changes in Resources Available

- Suppose that the availability of a resource changes.
- If the optimal production plan doesn't use all of a resource (the slack variable is positive) then an increase in the resource won't affect the optimal vector.
- A decrease in the resource by less than the value of the slack also won't affect the optimal vector.
- In the Dear Beer Co. problem, all of the slack variables are zero in the optimal tableau.
- So the optimal solution will be affected by a decrease in the availability of any of the three resources.
- It is possible to analyse the effect of changing the availability of more than one resource simultaneously but I'll only look at the case where the availability of a single resource is changed.

5.3.1 Changes in a Single Resource

Two Things to Remember about Matrix Multiplication

- Remember that the k^{th} column of the matrix AB is A times the k^{th} column of the matrix B.
- And that $\mathbf{y} = A\mathbf{x}$ (A an $m \times n$ matrix, $\mathbf{x}$ & $\mathbf{y}$ compatible vectors) means that
  - ⋆ **$\mathbf{y}$ can be written as** $x_1\mathbf{A_1} + x_2\mathbf{A_2} + \cdots + x_n\mathbf{A_n}$ where $\mathbf{A_1}, \ldots \mathbf{A_n}$ are the columns of the matrix A.
  - ⋆ So for example, if $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ then $A\mathbf{x}$ is just $\mathbf{A_1}$.

Now Back to Sensitivity Analysis

- Still using the Dear Beer Co. example, suppose that the amount of yeast available changes from 80 to $80 + a$ units, where a can be positive or negative.
- To see what happens, change 80 to $80 + a$ in the constant column of the original tableau giving a new tableau P_1:

$$P_1 = \begin{array}{|r|rrrrrrr|}
\hline
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 \\
\hline
0 & -6 & -5 & -3 & -7 & 0 & 0 & 0 \\
\hline
50 & 1 & 1 & 0 & 3 & 1 & 0 & 0 \\
150 & 2 & 1 & 2 & 1 & 0 & 1 & 0 \\
80 + a & 1 & 1 & 1 & 4 & 0 & 0 & 1 \\
\hline
\end{array}$$

- I showed on Slide 224 that once I know P and P^* it is easy to write down a "pivot matrix" Q such that $P^* = QP$.
- I asked you previously to check that

$$Q = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

- If I perform on P_1 the same sequence of pivots that transforms P into P^*, the resulting tableau is $P_1^* = QP_1$.

- As P_1^* differs from P^* only in the constant column, I'll just re-compute that column:

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 50 \\ 150 \\ 80+a \end{bmatrix} = \begin{bmatrix} 380+a \\ 40-2a \\ 30+a \\ 10+2a \end{bmatrix}.$$

- In fact I didn't need to do the full matrix-vector product as

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 50 \\ 150 \\ 80+a \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 50 \\ 150 \\ 80 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix}$$

$$= \begin{bmatrix} 380 \\ 40 \\ 30 \\ 10 \end{bmatrix} + a \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}$$

- So P_1^* is:

$$P_1^* = \begin{array}{|c|ccccccc|}
\hline
 & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_2} & \mathbf{s_3} \\
\hline
380 + a & 0 & 0 & 0 & 7 & 3 & 1 & 1 \\
\hline
40 - 2a & 1 & 0 & 0 & -4 & 1 & 1 & -2 \\
30 + a & 0 & 0 & 1 & 1 & -1 & 0 & 1 \\
10 + 2a & 0 & 1 & 0 & 7 & 0 & -1 & 2 \\
\hline
\end{array}$$

- This tableau is in optimal form provided that: $40 - 2a \geq 0, 30 + a \geq 0, 10 + 2a \geq 0$ which decodes to $-5 \leq a \leq 20$.
- For example if $a = 10$ extra units of yeast are available, then from P_1^* I can simply write down the new optimal vector:

$$\begin{bmatrix} 40 - 2a \\ 10 + 2a \\ 30 + a \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 40 \\ 0 \end{bmatrix}.$$

- And z decreases to -390 (income increases to 390). Remember the LP is a max problem.
- No surprise; extra resources should allow a better solution to be found. Remember the tableau represents a min problem but the "real" problem is a max problem.

- As I already said, very little matrix arithmetic is needed.
- I don't really need to use the pivot matrix Q to calculate the adjusted constant column entries in P_1^*.
- Just note that the coefficients of α in the constant column of P_1^* are the coefficients in the s_3 column of P^*.

- So I can read off from

$$P^* = \begin{array}{|c|ccccccc|}\hline & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 \\ \hline 380 & 0 & 0 & 0 & 7 & 3 & 1 & 1 \\ \hline 40 & 1 & 0 & 0 & -4 & 1 & 1 & -2 \\ 30 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \\ 10 & 0 & 1 & 0 & 7 & 0 & -1 & 2 \\ \hline \end{array}$$

with $s_3 = -10$ that the new optimal vector x is:

$$\begin{bmatrix} 40 + 2s_3 \\ 10 - 2s_3 \\ 30 - s_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 40 \\ 0 \end{bmatrix}.$$

- The rationale for taking $s_3 = -\alpha$ is simple;
  - ⋆ I noted in Section 3.3 that increasing the value of the slack variable for a resource is equivalent to decreasing the availability of that resource by the same amount.
    - ∗ If you like, **requiring** that extra resources are kept unused (increased slack) is equivalent to decreasing the availability of that resource.
  - ⋆ Similarly increasing the availability of a resource is equivalent to decreasing the value of the slack variable for the resource by the same amount.
  - ⋆ So requiring that s_3 becomes negative is equivalent to increasing the availability of resource 3 (adding to the value in the LH column of the starting tableau).

Another example (still using Dear Beer Co.)

- Suppose I want to analyse the effect of changing the availability of resource s_2 .
- I can skip the analysis using α (and Q).
- Just read off the coefficients in the s_2 column of P^* and **subtract** s_2 times each coefficient from the constant column of P^*.

- The constant column is now: $\begin{bmatrix} 40 - s_2 \\ 30 \\ 10 + s_2 \end{bmatrix}$.
- The constant column stays non-negative provided that $-10 \leq s_2 \leq 40$.
- If the availability (amount) of resource 2 is increased by 5, I compute the new optimal vector as:

$$\begin{bmatrix} 40 - s_2 \\ 10 + s_2 \\ 30 \\ 0 \end{bmatrix} = \begin{bmatrix} 45 \\ 5 \\ 30 \\ 0 \end{bmatrix} \quad \text{with } s_2 = -5.$$

- Check that z decreases to $-380 - 5 = -385$ so income increases to 385. (Why?)

Yet another example (still using Dear Beer Co.)

- This time suppose that I have a decrease of 25 in the amount of resource 2 available.
- I compute the new optimal vector as:

$$\begin{bmatrix} 40 - s_2 \\ 10 + s_2 \\ 30 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 35 \\ 30 \\ 0 \end{bmatrix} \quad \text{with } s_2 = 25.$$

- Check that z increases to $-380 + 25 = -355$ so income decreases to 355. (Why?)

A final note:

When in doubt use common sense to see that a reduction in resources available increases the optimal value of the objective function for a min problem and decreases it for a max problem.

This observation at least ensures that you won't "get the sign wrong".

5.4 Changes in Objective Coefficients (Costs)

- If you look again at the graphical solution of the T&C Corp problem, Fig.2, it is clear that changing the slope of the lines of constant objective value (contours) won't necessarily mean that a different optimal point will be selected.
- Can you say for what range of slopes the optimal point will not change?
- It isn't difficult to answer this question algebraically for "real problems" where graphical solutions are not possible.

5.4.1 Changes in Price for Basic Variables

- I'll use the Dear Beer Co. example yet again.
- Beer number 1 (variable x_1) is basic **in the optimal tableau.**
- Suppose I change the selling price for x_1 from €6 to €6+q, the initial tableau is:

$$P' = \begin{array}{|r|rrrrrrr|}
\hline
 & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_2} & \mathbf{s_3} \\
\hline
0 & -6-q & -5 & -3 & -7 & 0 & 0 & 0 \\
\hline
50 & 1 & 1 & 0 & 3 & 1 & 0 & 0 \\
150 & 2 & 1 & 2 & 1 & 0 & 1 & 0 \\
80 & 1 & 1 & 1 & 4 & 0 & 0 & 1 \\
\hline
\end{array}$$

- If I do the same pivots on P' as I did on P to get P^*, I get the tableau P''

$$P'' = \begin{array}{|c|ccccccc|} \hline & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 \\ \hline 380 & -q & 0 & 0 & 7 & 3 & 1 & 1 \\ \hline 40 & 1 & 0 & 0 & -4 & 1 & 1 & -2 \\ 30 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \\ 10 & 0 & 1 & 0 & 7 & 0 & -1 & 2 \\ \hline \end{array}$$

- You can check this by multiplying P' by the pivot matrix Q — the second column of P^* (the x_1 column) is the only column to change and because of the zeros in the first column of Q, only the first row of P^* changes.

- In fact, P'' differs from P^* only in the c_1 cost entry.

- The matrix is not quite in optimal form. (Why?)

- To pivot P'' into optimal form, pivot on the circled element:

$$P''' = \begin{array}{|c|ccccccc|}
\hline
 & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_2} & \mathbf{s_3} \\
\hline
380+40q & \mathbf{0} & 0 & 0 & 7-4q & 3+q & 1+q & 1-2q \\
\hline
40 & 1 & 0 & 0 & -4 & 1 & 1 & -2 \\
30 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \\
10 & 0 & 1 & 0 & 7 & 0 & -1 & 2 \\
\hline
\end{array}$$

- The tableau P''' is in optimal form if

$$7 - 4q \geq 0, 3 + q \geq 0, 1 + q \geq 0, 1 - 2q \geq 0.$$

- Or just,

$$-1 = \max\left\{\frac{-3}{1}, \frac{-1}{1}\right\} \leq q \leq \min\left\{\frac{7}{4}, \frac{1}{2}\right\} = \frac{1}{2}.$$

- In fact, I can calculate the range for q directly from P^*.
- Here it is again:

$$P^* = \begin{array}{|c|ccccccc|} \hline & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_2} & \mathbf{s_3} \\ \hline 380 & 0 & 0 & 0 & 7 & 3 & 1 & 1 \\ \hline 40 & 1 & 0 & 0 & -4 & 1 & 1 & -2 \\ 30 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \\ 10 & 0 & 1 & 0 & 7 & 0 & -1 & 2 \\ \hline \end{array}$$

- It's easy to see that the coefficient of q in the change in the selling price (the negative of the objective value) is just the optimal value for the basic variable x_1 (40).
- You can also see that the coefficients for the non-basic variables in the row that defines x_1 $(-4, 1, 1, -2)$ are exactly the coefficients that multiply q in the changed objective coefficients in P'''.

- For each nonbasic column in P^*, if I divide the objective function coefficient by **minus** the entry in the x_1 row, I get the ratios in the range on q.
- The general rule is:
  - ⋆ If the change q in the selling price of a product corresponding to the basic variable in row r of P^* satisfies:

$$\max_{k \in \text{non-basic cols}} \left\{ \frac{-c_k}{a_{rk}} \middle| a_{rk} > 0 \right\} \leq q$$
$$\leq \min_{k \in \text{non-basic cols}} \left\{ \frac{-c_k}{a_{rk}} \middle| a_{rk} < 0 \right\}$$

  - ⋆ then the optimal vector does not change
  - ⋆ and the optimal value changes by qx_k^* where x_k^* is the value in the optimal tableau of the variable whose selling price is being changed.
- Again, it is easier to "see" the solution than use a rule.

5.4.2 Changes in Price for Nonbasic Variables

- If the selling price for a nonbasic (in the optimal tableau) variable changes, the analysis is even simpler.
- For example, if the sales price of the beer corresponding to variable x_4 changes from €7 to €7+r, then the cost coefficient in the x_4 column of the starting tableau P becomes $(-7-r)$.
- Performing the pivots to get P^* (multiplying by Q) causes this coefficient to change to $(7-r)$, **the only change in P^*** is that the x_4 column becomes:

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -7-r \\ 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 1 \\ 7 \end{bmatrix} - r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7-r \\ -4 \\ 1 \\ 7 \end{bmatrix}.$$

- So the new tableau is in optimal form provided $7 - r \geq 0$ — so the optimal vector doesn't change unless the selling price for beer #4 is increased beyond €14.
- In general, a little matrix algebra shows that the selected column (x_k — say) in the starting tableau will **always** be transformed to the same column in P^* with $-r$ added to the cost coefficient (or $-q$ for a basic variable).
- For a basic variable, if $q > 0$ then pivoting is needed which complicates things, as I showed.
- For a nonbasic variable, if the cost coefficient in the optimal tableau is positive, then the selling price may be increased by an amount up to that value (which is reasonable).
- Finally, because no pivoting is needed, changes in the the selling price for a nonbasic (in the optimal tableau) variable have no effect on the optimal objective value.

5.5 Exercises for Chapter 5

1. Consider the following optimal form tableau for a LP:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-75	0	4	0	3	2	0	0
10	1	-1	0	-2	-1	0	0
5	0	1	0	2	3	0	1
20	0	-2	1	-1	1	0	0
30	0	1	0	-1	0	1	0

   (a) What is the optimal solution?

   (b) Find an optimal vector when the additional requirement $x_2 = 3$ is added to the original model.

   (c) If I need $x_2 = 6$, is the LP still feasible?

(d) Using the second constraint equation, find upper bounds on the variables x_2, x_4, x_5, x_7 that must be satisfied by any feasible solution.

(e) Find an optimal vector when the additional requirement $x_1 = 12$ is added to the original model.

(f) Find an optimal vector when the additional requirement $x_1 = 21$ is added to the original model — or explain why this is impossible.

2. The following tableau P is the initial canonical form tableau for a resource allocation problem of the form
$\max \mathbf{c}^\mathsf{T}\mathbf{x}$ such that $A\mathbf{x} \le \mathbf{b}$ with $\mathbf{x} \ge 0$:

$\mathsf{P} =$

	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$
0	−7	−8	−3	−7	0	0	0
50	1	2	1	1	1	0	0
40	2	2	1	1	0	1	0
30	1	2	1	5	0	0	1

The optimal tableau is:

$$P^* = \begin{array}{|c|ccccccc|}
\hline
 & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_2} & \mathbf{s_3} \\
\hline
150 & 0 & 0 & 1 & 1 & 0 & 3 & 1 \\
\hline
20 & 0 & 0 & 0 & -4 & 1 & 0 & -1 \\
10 & 1 & 0 & 0 & -4 & 0 & 1 & -1 \\
10 & 0 & 1 & \frac{1}{2} & 9/2 & 0 & -\frac{1}{2} & 1 \\
\hline
\end{array}$$

(a) Write down an optimal vector for the dual problem.

(b) Find a matrix Q s.t. $P^* = QP$.

(c) What is the minimum amount that I should charge for 8 units of resource 2 so that the total revenue from selling these 8 units and from selling products is at least €150?

(d) What is the minimum amount that I should charge for 22 units of resource 2 so that the total revenue from selling these 22 units and from selling products is at least €150? (Check your answer by adding the constraint $s_2 \geq 22$ to P^* and pivoting to optimality.)

(e) What is the new optimal vector x^* if I need to make 10 units of product 3?

(f) What is the new optimal vector x^* if I need to make 25 units of product 3?

(g) What is the new optimal vector x^* if only 25 units of resource 3 are available instead of the original 30?

(h) What is the new optimal vector x^* if 35 units of resource 3 are available instead of the original 30?

(i) Give a range in which the selling price of product 1 can vary without changing the optimal vector/basic feasible point .

3. The following tableau P is the initial canonical form tableau for a resource allocation problem of the form
$\max \mathbf{c}^{\mathsf{T}}\mathbf{x}$ such that $A\mathbf{x} \le \mathbf{b}$ with $\mathbf{x} \ge 0$:

$$P = \begin{array}{|r|rrrrrrr|}
\hline
 & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_2} & \mathbf{s_3} \\
\hline
0 & -6 & -1 & -4 & -5 & 0 & 0 & 0 \\
\hline
20 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
10 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\
5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

The optimal tableau is:

$$P^* = \begin{array}{|c|ccccccc|} \hline & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{s_1} & \mathbf{s_2} & \mathbf{s_3} \\ \hline 55 & 0 & 0 & 7 & 0 & 0 & 5 & 1 \\ \hline 5 & 0 & 0 & -2 & 0 & 1 & -1 & -1 \\ 5 & 0 & -1 & 1 & 1 & 0 & 1 & -1 \\ 5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

(a) Find a matrix Q s.t. $P^* = QP$.

(b) What is the minimum amount that I should charge for 8 units of resource 2 so that the total revenue from selling these 8 units and from selling products is at least €55?

(c) What is the new optimal vector x^* if only 4 units of resource 3 are available instead of the original 5?

(d) What is the new optimal vector x^* if 8 units of resource 3 are available instead of the original 5?

(e) Give a range in which the selling price of product 1 can vary without changing the optimal vector/basic feasible point .

(f) What is the smallest amount by which the selling price of product 3 should be increased so that product 3 would be produced in an optimal production plan?

(g) What is the new optimal production plan if the selling price of product 4 increases to €6?

4. For the Dear Beer Co. problem, (re-stated at the start of Section 5.1) answer the following questions:
   (a) What is the new optimal production plan if only 40 units of malt (the first resource) are available instead of the original 50 units?
   (b) What is the new optimal production plan if an extra 15 units of malt (the first resource) become available?
   (c) What is the new optimal production plan if only 5 units of malt (the first resource) are available?