UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Department of Mathematics & Statistics
Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4303

SEMESTER: Spring 2017

MODULE TITLE: Operations Research 1

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

GRADING SCHEME:

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES:

- **Answer four of the five questions correctly for full marks** 70%**.**
- **For convenience, each question is assigned 25 marks.**
- **Ruled paper for recording transportation tableaux is appended to this Examination paper.**
- **Your examination paper <u>must</u> be returned with your script.**

1 The manager of a large restaurant is deciding how to source tablecloths for each of the 7 days that the restaurant is open each week (Monday–Sunday).

- She can buy new tablecloths for €10 each.
- She must buy (pre-purchase) enough before the week's business starts (early on Monday morning) to cover the week's requirements.
- After being used on a single day the tablecloths must be sent for cleaning.
- They can either be sent to SlowClean (2–day service) or FastClean (1–day service).
  - ⋆ For example, cloths sent to FastClean on Monday are available for use on Wednesday
  - ⋆ and cloths sent to SlowClean on Monday are available for use on Thursday.
-
  - ⋆ SlowClean charge €1 per cloth
  - ⋆ FastClean charge €3 per cloth
- For the 7 days, Monday–Sunday, the restaurant needs to have the following numbers of cloths available for use:

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
n_1	n_2	n_3	n_4	n_5	n_6	n_7
110	100	160	120	180	200	120

  (For convenience, just refer to the number of cloths needed each day as $n_1, \dots, n_7$ in your answers to the questions below.)
- At the end of the week (late Sunday night) all cloths are sold for €2 each as they cannot be re-used in the following week.

Formulate a LP model for this problem that will determine the pre-purchase/cleaning (combination of pre-purchase/SlowClean/FastClean) plan with minimum cost as follows:

(a) Make a choice of decision variables for the problem. 2

Hint(s): one natural choice of decision variable is (p_0, say) the number of cloths to "Pre-Purchase" before start of business on Monday. You will probably need 9 others related to the number of cloths to send to the two cleaners on each of the days when it makes sense to do so. (For example there is no point in sending cloths to FastClean on Saturday or to SlowClean on Friday.)

Marks will only be awarded for a detailed (and correct) explanation of your choice.

(b) Make a choice of "intermediate" variables, i.e. variables that are useful when tracking the day-to-day variation of the problem. Again, marks will only be awarded for a detailed (and correct) explanation of your choice and only if the intermediate variables are expressed in terms of the decision variables. 8

(c) (i) List the constraints (in algebraic notation) for your problem in terms of the decision and intermediate variables as appropriate. 4

(ii) Re-phrase all the constraints in terms of the decision variables only. 6

(d) (i) State the objective function z (in algebraic notation) in terms of the decision and intermediate variables as appropriate. 3

(ii) Re-phrase z in terms of the decision variables only. 2

2 (a) Consider the following LP:

$$\begin{aligned} \max\ & 2x_1 - 3x_2 \\ \text{subject to}\quad & x_1 + x_2 \geq 3 \\ & x_1 - 2x_2 \leq 7 \end{aligned}$$

with $x_1, x_2 \geq 0$.

(i) Reformulate the LP into a Standard Form (SF) LP. 1

(ii) Write a Simplex Tableau for your SF LP. 2

(iii) Explain why the tableau is not in Canonical Form (CF). 1

(iv) Use the Dual Simplex method (DSM) to pivot to CF. 4

(v) Explain why the tableau in not in Optimal Form (OF). 1

(vi) Perform one Simplex Method (SM) pivot. 4

(vii) Explain carefully why no further progress is possible (no further pivots can be performed). 1

(viii) Change the signs of the elements in rows 2 & 3 of column 3 of your tableau. Which element should you pivot on to apply a Simplex Method (SM) pivot. (**Do not perform the pivot.**) 1

(ix) The tableau is now:

46/3	0	0	1/3	7/3
13/3	1	0	−2/3	1/3
4/3	0	1	1/3	1/3

What are the the optimal values of x_1, x_2 and the objective function z for the max problem? 1

(Q.2 is continued on the next page.)

(b) As part of the process of formulating an LP in SF, "free variables" must be expressed in terms of non-negative variables.

(i) Starting with a LP that is otherwise in SF, explain carefully how free variables can be eliminated using the expression $\mathbf{x} = \mathbf{y} - w\mathbf{e}$ where $\mathbf{e}$ is a constant vector of ones, $\mathbf{x}$ is a vector of free variables, $\mathbf{y}$ is a non-negative vector and w is a non-negative scalar variable. 5

(ii) Illustrate your answer by reformatting your starting LP in part (a)(ii) of this Question with the non-negativity conditions dropped. Write the extra column of the tableau. 3

3 (a) Start with the standard dual pair in Appendix C and show (using steps (i)–(v) below) that the optimal solutions to both are equal:

(i) Introduce a vector $\mathbf{s}$ of slacks $s_1, \dots, s_m$ (where m is the number of constraints) into the min problem and show that the corresponding Simplex Tableau takes the form

$$P = \begin{array}{|c|c|c|} & \mathbf{x} & \mathbf{s} \\ 0 & \mathbf{c}^T & \mathbf{0}^T \\ \mathbf{-b} & -A & \mathbf{I} \end{array}.$$

3

(ii) Let P be the simplex tableau found in part (i) and P^* the OF tableau, say

$$P^* = \begin{array}{|c|c|c|} & \mathbf{x} & \mathbf{s} \\ -d & \mathbf{u}^T & \mathbf{v}^T \\ \mathbf{b}^* & D & B \end{array}$$

with optimal vector x^* and $d = \mathbf{c}^T x^*$. Recall that the pivot matrix Q is the result of applying the same pivots to I_{m+1} as were applied to P in order to pivot it to OF and that $P^* = QP$.

Show that $Q = \begin{bmatrix} 1 & \mathbf{v}^T \\ \mathbf{0} & B \end{bmatrix}$ where $\begin{bmatrix} \mathbf{v}^T \\ B \end{bmatrix}$ are the right-hand m columns of P^* corresponding to the m slack variables. 6

(iii) Now use the fact that $P^* = QP$ to show that $\mathbf{v} \geq \mathbf{0}$, $A^T\mathbf{v} \leq \mathbf{c}$ and $d = \mathbf{v}^T\mathbf{b}$. 5

(iv) Show (without using tableaux) for any primal feasible $\mathbf{x}$ and any dual feasible $\mathbf{y}$ that $\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y}$. 2

(v) Finally explain why you can conclude that $c^T x^* = b^T\mathbf{v}$ and that $\mathbf{v}$ is the optimal vector for the dual problem. 3

(b) Show that the dual of an LP in SF is 6

$$\begin{aligned} \max \quad & \mathbf{b}^T \mathbf{y} \\ \text{subject to} \quad & A^T \mathbf{y} \le \mathbf{c} \\ & \mathbf{y} \quad \text{free.} \end{aligned} \qquad (1)$$

4 (a) Consider the optimal form (OF) tableau

$T =$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
10	0	11	0	0	0	8	1
13	0	0	0	0	1	0	0
59	0	2	1	0	0	1	7
21	0	1	0	1	0	0	−3
40	1	−1	0	0	0	10	−4

(i) Find (without re-solving the LP) the optimal vector and objective value when the additional requirement $x_6 = 2$ is added to the problem. 2

(ii) Suppose now that the condition $x_6 = 6$ must be satisfied.

A. Explain why the tableau T must first be pivoted so that x_6 is increased up to its minimum row ratio for the tableau. 1

B. In what row & column must the pivot be performed? 1

(iii) Given that the result of the pivot is

$T_1 =$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
−22	−4/5	59/5	0	0	0	0	21/5
13	0	0	0	0	1	0	0
55	−1/10	21/10	1	0	0	0	37/5
21	0	1	0	1	0	0	−3
4	1/10	−1/10	0	0	0	1	−2/5

state which of the currently non-basic variables can be increased from zero (so that x_6 is increased to 6) noting the corresponding increase in z? 2

(iv) Determine which choice of variables to increase is "cheaper" and find the optimal vector and objective value corresponding to $x_6 = 6$. 4

(Q.4 is continued on the next page.)

(b) The following tableau T is the initial canonical form tableau for a resource allocation problem of the form max $\mathbf{c}^T\mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ with $\mathbf{x} \geq 0$:

$T =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
0	−6	−3	−1	−2	0	0	0
10	5	1	2	1	1	0	0
15	1	1	2	2	0	1	0
5	1	1	2	1	0	0	1

with optimal form (OF) tableau

$T^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
75/4	0	0	5	1	3/4	0	9/4
5/4	1	0	0	0	1/4	0	−1/4
10	0	0	0	1	0	1	−1
15/4	0	1	2	1	−1/4	0	5/4

(i) Write the matrix Q such that $T^* = QT$. (See Q.3(a)(ii) above.) 2

(ii) Explain using Q what the effect on T^* is of adding a (positive or negative) to the inital resources available for resource 3. 3

(iii) Find the new optimal vector and objective if 4 units of resource 3 are available instead of the original 5. 3

(c) Suppose, for a resource allocation problem as in part (b), that the price in the starting canonical form of a variable x_i that is basic in the optimal tableau is changed by the addition of q (positive or negative) to the price.

(i) Show that the effect on an optimal tableau is to add q times the row in T^* associated with x_i to the top (objective) row of T^* so only z changes and not the optimal x. 3

(ii) What is the maximum amount by which the price associated with the variable x_2 in T in part (b) above may be **increased** or **decreased** while keeping the currently basic variables basic? 2

(iii) Suppose that the price associated with variable x_1 in the tableau T in part (b) is increased from €6 to €8. What is the effect on the objective function z? 2

5 The (balanced) Transportation Problem

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \text{ for } i = 1, \dots, m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \text{ for } j = 1, \dots, n.$$

$$x_{ij} \geq 0, \text{ for each } i \text{ and } j.$$

with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} = b_j$ can be written as an LP in Standard Form:

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

referring to Appendix E for the definitions of A, **b** and **c**. (Note that **b** is **not** the vector of demands $b_1, \dots, b_n$.)

(a) (i) Use the expression for the dual of an LP in SF in Eq. 1 in Q3 (b) above to show that the dual of the Transportation Problem is 7

$$\max \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

subject to

$$u_i + v_j \leq c_{ij}, \quad i = 1, \dots, m, \quad j = 1, \dots, n$$

u&**v** free.

(ii) Explain briefly, without further algebra but referring to the result (established in Q.3 (a)) that the optimal objective values for a primal-dual pair are equal, how this result is used to determine whether a transportation tableau is optimal. 2

(b) A company has four manufacturing plants, all making the same product. There are three outlets that require the product. Unfortunately, some of the plants can only ship to certain destinations.

- Plants 1, 2 and 3 produce 10, 15, 20 & 15 items respectively per day.
- Plant 1 can ship to all three outlets with unit shipping costs of 8, 6, and 9 respectively.
- Plant 2 can only ship to outlets 2 & 3 with unit shipping costs of 4 & 1 respectively.
- Plant 3 can ship to all three outlets with unit shipping costs of 5, 1 &2 respectively.
- Plant 4 can ship to all three outlets with unit shipping costs of 3, 5 & 7 respectively.
- The three outlets require 5, 25 and 15 items per day respectively. There is an excess in production capacity.

(i) Formulate this problem as a transportation problem by writing a transportation tableau. (Note that supply does not equal demand so a dummy demand with zero shipping costs is needed. Assign a "large" positive cost N to the forbidden link .) 2

(ii) Use the NW Corner Method (NWCM) to find an initial feasible tableau. 2

(iii) Use the SCEM method to find an initial feasible tableau. 2

(iv) Calculate the costs of the two starting solutions. Which is better? 1

(c) Using either of your two starting tableau and referring to the statement of the Transportation Algorithm in App. D if you wish:

(i) Calculate dual variables u and v. 1

(ii) "Adjust costs" replacing the costs in non-basic positions by $c_{ij} - u_i - v_j$. 1

(iii) Is the tableau optimal? Explain why or why not. 1/2

(d) Your tableau at this stage may vary depending on your choices in SCEM so take the following tableau as your current tableau for the next part of the question.

11	8	8	0^{10}
$N+3$	6	0^{10}	0^{5}
5	0^{20}	-2	-3
0^{5}	0^{5}	0^{10}	-6

(i) Find the "loop" connecting a succession of basic positions, starting at the non-basic position with most negative cost. 1

(ii) Make the max allowable increase/decrease t in x_{ij} for the basic positions in the loop. 1

(iii) Update your tableau — changing the basic variables and updating the assignments as necessary. 1

(e) The following tableau is calculated at a later iteration:

5	2	4	0^{10}
$N+1$	4	0^{15}	4
5	0^{20}	0^{0}	3
0^{5}	0^{5}	2	0^{5}

(i) Explain why it is optimal. $\frac{1}{2}$

(ii) What is the cost associated with this optimal tableau? 1

(iii) Draw a network diagram (arrow diagram) displaying your solution. 1

(iv) Which producer(s) have excess stock and how much? 1

Appendix of Results

A **Algorithm 1 (Simplex Method)**

begin (Start with a Canonical tableau s.t. $\mathbf{b} \geq \mathbf{0}$.)
while NOT finished do
if $c_j \geq 0$ for all j
then **STOP** (Tableau is optimal.)
else Select j s.t. $c_j < 0$.
fi
if $a_{ij} \leq 0$ for all $i = 1, \ldots, m$
then **STOP** (Problem is unbounded.)
fi
Select k such that:
$\frac{b_k}{a_{kj}} = \min_i \left\{ \frac{b_i}{a_{ij}} \text{such that } a_{ij} > 0 \right\}$ (k attains the min.)
Pivot on a_{kj}. (Divide Row k across by a_{kj} and add
end ... multiples of Row k to the rows above & below
end ... ... introducing zeros into column j.)

B **Algorithm 2 (Dual Simplex Method)**

begin (Start with a tableau s.t. $\mathbf{c} \geq \mathbf{0}$.)
while NOT finished do
if $b_i \geq 0$ for all i
then **STOP** (Tableau is optimal.)
else Select i s.t. $b_i < 0$.
fi
if $-a_{ij} \leq 0$ for all $j = 1, \ldots, n$
then **STOP** (Dual unbounded $\equiv$ Primal infeasible.)
fi
Select k such that:
$\frac{c_k}{a_{ik}} = \max_j \left\{ \frac{c_j}{a_{ij}} \text{such that } a_{ij} < 0 \right\}$ (k attains max.)
Pivot on a_{ik}. (Divide Row k across by a_{ik} and add
end ... multiples of Row k to the rows above & below
end ... ... introducing zeros into column i.)

C The following pair of LP's are the **standard dual pair**:

$$
\begin{array}{rl}
\min & \mathbf{c}^T\mathbf{x} \\
\text{subject to} & A\mathbf{x} \geq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{array}
\qquad
\begin{array}{rl}
\max & \mathbf{b}^T\mathbf{y} \\
\text{subject to} & A^T\mathbf{y} \leq \mathbf{c} \\
& \mathbf{y} \geq 0.
\end{array}
$$

D Transportation Algorithm

(a) Find an initial basic feasible point $\mathbf{x}$.

(b)
- For the current solution $\mathbf{x}$ and the current cost coefficients c_{ij}, find a dual vector $(\mathbf{u}, \mathbf{v})$ such that $u_i + v_j = c_{ij}$ for all basic positions (i, j).
- Calculate the adjusted cost coefficients (a.c.c.) for all positions (i, j).

(c)
- If each a.c.c is ≥ 0, STOP.
- ELSE Pick the position with the most negative a.c.c and find the unique loop starting there (with all other positions basic).

(d)
- Shift as much as possible around the loop to get a new basic feasible point $\mathbf{x}$.
- GOTO Step 2 with the a.c.c's as the new costs.

E Notation for the Transportation Problem written as a LP in matrix notation.

- I_n is the $n \times n$ identity matrix.
- $e_n^T = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$ (n ones).
- $z_n^T = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}$ (n zeros).
- A is the $(m+n) \times (mn)$ matrix $A = \begin{bmatrix} e_n^T & z_n^T & z_n^T & \dots & z_n^T & z_n^T \\ z_n^T & e_n^T & z_n^T & \dots & z_n^T & z_n^T \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ z_n^T & z_n^T & z_n^T & \dots & e_n^T & z_n^T \\ z_n^T & z_n^T & z_n^T & \dots & z_n^T & e_n^T \\ I_n & I_n & I_n & \dots & I_n & I_n \end{bmatrix}$.
- $\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.
- The cost coefficients $\mathbf{c}$ are just the column vector of length mn:

$$\mathbf{c} = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1n} \\ c_{21} \\ c_{22} \\ \vdots \\ c_{2n} \\ \vdots \\ c_{m1} \\ c_{m2} \\ \vdots \\ c_{mn} \end{bmatrix}.$$

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