UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Department of Mathematics & Statistics
Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4303

SEMESTER: Spring 2016

MODULE TITLE: Operations Research 1

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

GRADING SCHEME:

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES:

- **Answer four of the five questions correctly for full marks 70%.**
- **For convenience, each question is assigned 25 marks.**
- **Ruled paper for recording transportation tableaux is appended to this Examination paper.**
- **Your examination paper must be returned with your script.**

1 Gift Grub Co. produce two food products, BatterBurgers & SpiceBurgers. The main ingredients are Beef and Onions.

- The total cost (purchase and processing) per unit (1 kg.) of Beef used is €20 and the total cost per unit (1 kg.) of Onions used is €5.
- The unit sales price of BatterBurgers is €20 and the unit sales price of SpiceBurgers is €15.
- Each unit of BatterBurgers produced requires 0.5 units of Beef and 0.5 units of Onions.
- Each unit of SpiceBurgers produced requires 0.2 units of Beef and 0.8 units of Onions.
- Daily demand for BatterBurgers lies between 10 & 15 units.
- Daily demand for SpiceBurgers lies between 12 & 20 units.
- Daily supply of Beef cannot exceed 10 units.
- Daily supply of Onions cannot exceed 20 units.

(a) Explain carefully in algebraic notation (not just in words):

(i) what are the decision variables? 2

(ii) what are the other variables in the problem (expressed in terms of the decision variables)? 4

(iii) what is the objective function (expressed in terms of the decision variables)? 2

(iv) is the problem a max or min problem? 1

(v) what are the constraints (expressed in terms of the decision variables)? 2+2+2+2

(b) Formulate the problem as a Standard Form (SF) Linear Program (LP). 1

(c) Write a Simplex Tableau for your SF LP. 4

(d) Explain whether the tableau is in Canonical Form. 1

(e) What is the first pivot (specify row & column) that needs to be applied? Explain why. (**Do not apply the pivot.**) 2

2 (a) Consider the following LP:

$$\max x_1 + 2x_2$$
$$\text{subject to} \quad x_1 + x_2 \geq 3$$
$$x_1 - 2x_2 \leq 7$$

with $x_1, x_2 \geq 0$.

(i) Reformulate the LP into a Standard Form (SF) LP. 1

(ii) Write a Simplex Tableau for your SF LP. 2

(iii) Explain why the tableau is not in Canonical Form (CF). 1

(iv) Use the Dual Simplex method (DSM) to pivot to CF. 4

(v) Explain why the tableau in not in Optimal Form (OF). 1

(vi) Explain why the tableau is in Unbounded Form (UF). 1

(vii) Change the signs of all the elements in the unbounded column (including the objective coefficient) of your tableau to positive so that the tableau is no longer in UF. Then set the sign of the objective coefficient for x_1 to negative and perform one Simplex Method (SM) pivot. 3

(viii) The tableau will now be in OF. What are the the optimal values of x_1, x_2 and z? 1

(b) Consider the following Simplex Tableau T:

T =

	x_1	x_2	x_3	x_4	x_5	x_6
5	0	b	e	0	0	0
1	0	d	0	1	0	0
2	0	-3	2	0	1	1
a	1	c	1	0	0	1

State conditions that the parameters a, b, c, d and e must satisfy so that

(i) T is in OF. 1

(ii) T is in unbounded form. 1

(iii) T is in one of the two infeasible forms. 1

(c) As part of the process of formulating an LP in SF, "free variables" must be expressed in terms of non-negative variables.

(i) Starting with a LP that is otherwise in SF, explain carefully how free variables can be eliminated using the expression $\mathbf{x} = \mathbf{y} - w\mathbf{e}$ where $\mathbf{e}$ is a constant vector of ones, $\mathbf{x}$ is a vector of free variables, $\mathbf{y}$ is a non-negative vector and w is a non-negative scalar variable. 5

(ii) Illustrate your answer by reformatting your starting LP in part (a) with the non-negativity conditions dropped. Write the extra column of the tableau. 3

3 (a) Start with the standard dual pair in Appendix C and show (using steps (i)–(v) below) that the optimal solutions to both are equal:

(i) Introduce slacks **s** into the min problem and show that the corresponding Simplex Tableau takes the form

$$\mathrm{P} = \begin{array}{|c|c|c|} \hline & \mathbf{x} & \mathbf{s} \\ \hline 0 & \mathbf{c}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} \\ \mathbf{-b} & \text{-A} & \mathbf{I} \\ \hline \end{array}.$$

3

(ii) Show that if P is a simplex tableau and if P^* is the OF tableau, say

$$\mathrm{P}^* = \begin{array}{|c|c|c|} \hline & \mathbf{x} & \mathbf{s} \\ \hline \text{-d} & \mathbf{u}^{\mathsf{T}} & \mathbf{v}^{\mathsf{T}} \\ \mathbf{b}^* & \mathrm{D} & \mathrm{B} \\ \hline \end{array}$$

with optimal vector x^* and $d = \mathbf{c}^{\mathsf{T}}x^*$ then there is a matrix Q (the pivot matrix) such that $\mathrm{P}^* = \mathrm{QP}$ where $\mathrm{Q} = \begin{bmatrix} 1 & \mathbf{v}^{\mathsf{T}} \\ \mathbf{0} & \mathrm{B} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{v}^{\mathsf{T}} \\ \mathrm{B} \end{bmatrix}$ are the right-hand m columns of P^*. 6

(iii) Now use the fact that $\mathrm{P}^* = \mathrm{QP}$ to show that $\mathbf{v} \geq \mathbf{0}$, $\mathrm{A}^{\mathsf{T}}\mathbf{v} \leq \mathbf{c}$ and $d = \mathbf{v}^{\mathsf{T}}\mathbf{b}$. 5

(iv) Show (without using tableaux) for any primal feasible **x** and any dual feasible **y** that $\mathbf{c}^{\mathsf{T}}\mathbf{x} \geq \mathbf{b}^{\mathsf{T}}\mathbf{y}$. 2

(v) Finally explain why you can conclude that $c^{\mathsf{T}}x^* = b^{\mathsf{T}}\mathbf{v}$ and that **v** is the optimal vector for the dual problem. 3

(b) Show that the dual of an LP in SF is 6

$$\begin{aligned} \max \quad & \mathbf{b}^{\mathsf{T}}\mathbf{y} \\ \text{subject to} \quad & \mathrm{A}^{\mathsf{T}}\mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \quad \text{free.} \end{aligned} \qquad (1)$$

4 (a) Consider the optimal form (OF) tableau

$T =$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
−15	0	0	3	7	2	0	0
45	1	0	−2	5	1	0	0
15	0	0	2	−3	−3	1	0
18	0	1	−1	3	1	0	0
30	0	0	−1	−2	10	0	1

(i) Find (without re-solving the LP) the optimal vector and objective value when the additional requirement $x_5 = 1$ is added to the problem. 2

(ii) Suppose now that the condition $x_5 = 4$ must satisfied. Explain why the tableau T must first be pivoted so that x_5 is increased up to its minimum row ratio for the tableau. 2

(iii) Given that the result of the pivot is

$T_1 =$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
−21	0	0	16/5	37/5	0	0	−1/5
42	1	0	−19/10	26/5	0	0	−1/10
24	0	0	17/10	−18/5	0	1	3/10
15	0	1	−9/10	16/5	0	0	−1/10
3	0	0	−1/10	−1/5	1	0	1/10

which currently non-basic variable should be increased from zero with minimal increase in z? 3

(iv) Using the "cheaper" choice, find the optimal vector and objective value corresponding to $x_5 = 4$. 3

(b) The following tableau T is the initial canonical form tableau for a resource allocation problem of the form max $\mathbf{c}^T\mathbf{x}$ such that $A\mathbf{x} \le \mathbf{b}$ with $\mathbf{x} \ge 0$:

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
0	−9	−5	−15	−3	0	0	0
50	5	1	2	1	1	0	0
20	1	1	2	2	0	1	0
60	1	1	2	1	0	0	1

with optimal form (OF) tableau

$T^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
645/4	0	5/2	0	93/8	3/8	57/8	0
15/2	1	0	0	−1/4	1/4	−1/4	0
25/4	0	1/2	1	9/8	−1/8	5/8	0
40	0	0	0	−1	0	−1	1

(i) Write the matrix Q such that $T^* = QT$. (See Q.3(a)(ii) above.) 2

(ii) Explain using Q what the effect on T^* is of adding a (positive or negative) to the inital resources available for resource 2. 3

(iii) Find the new optimal vector and objective if 25 units of resource 2 are available instead of the original 20. 3

(c) Suppose, for a resource allocation problem as in part (b), that the price in the starting canonical form of a variable x_i that is basic in the optimal tableau is changed by the addition of q (positive or negative) to the price.

(i) Show that the effect on the optimal tableau is to add q times the row in T^* associated with x_i to the top (objective) row of T^* so only z changes and not the optimal x. 3

(ii) Suppose that the price associated with variable x_3 in the tableau T in part (b) is increased from €15 to €16. What is the effect on the objective function z? 2

(iii) What is the maximum amount by which the price associated with variable x_3 may be **increased** while keeping the currently basic variables basic? 2

5 The (balanced) Transportation Problem

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i, \text{ for } i = 1, \ldots, m.$$
$$\sum_{i=1}^{m} x_{ij} = b_j, \text{ for } j = 1, \ldots, n.$$
$$x_{ij} \geq 0, \text{ for each } i \text{ and } j.$$

with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} = b_j$ can be written as an LP in Standard Form:

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

referring to Appendix E for the definitions of A, **b** and **c**. (Note that **b** is **not** the vector of demands $b_1, \ldots, b_n$.)

(a) (i) Use the expression for the dual of an LP in SF in Eq. 1 in Q3 (b) above to show that the dual of the Transportation Problem is 6

$$\max \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

subject to

$$u_i + v_j \leq c_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

u&**v** free.

(ii) Explain briefly, without further algebra but referring to the result (established in Q.3 (a)) that the optimal objective values for a primal-dual pair are equal, how this result is used to determine whether a transportation tableau is optimal. 2

(b) A company has three manufacturing plants, all making the same product. per day. There are four outlets that require the product. Unfortunately, some of the plants can only ship to certain destinations.

- Plants 1, 2 and 3 produce 15, 15 & 10 items respectively.
- Plant 1 can ship to all four outlets with unit shipping costs of 2, 5, 4 and 3 respectively.
- Plant 2 can ship to all four outlets with unit shipping costs of 7, 2, 1 and 6 respectively.
- Plant 3 can only ship to outlets 2, 3 & 4 with unit shipping costs of 2, 3 & 5 respectively.
- The four outlets require 10, 20, 5 and 10 items per day respectively. There is a shortfall in production capacity.

(i) Formulate this problem as a transportation problem by writing a transportation tableau. (Note that supply does not equal demand so a dummy supply with zero shipping costs is needed. Assign a "large" positive cost N to the forbidden link .) 2

(ii) Use the NW Corner Method (NWCM) to find an initial basic feasible solution. 2

(iii) Use the SCEM method to find an initial basic feasible solution. **N.B. Make choices that avoid the cell containing N becoming basic.** 2

(iv) Which starting solution is better? Explain. 1

(v) Starting with your SCEM starting tableau and referring to the statement of the Transportation Algorithm in App. D if you wish:

A. Calculate dual variables u and v. 1

B. "Adjust costs" replacing the costs in non-basic positions by $c_{ij} - u_i - v_j$. 1

C. Is the tableau optimal? Explain why or why not. 1/2

(vi) If the NWCM is used to generate a starting tableau, several iterations of the Transportation algorithm are needed. At one of these iterations the following tableau is generated. The cost associated with this tableau (using the starting costs) is 90.

0^{10}	5	3	0^{5}
3	0^{15}	-2	1
$N-4$	0^{5}	0^{5}	0^{0}
1	3	2	0^{5}

A. Find the "loop" connecting a succession of basic positions, starting at the non-basic position with most negative cost. 1

B. Make the max allowable increase/decrease in x_{ij} for the basic positions in the loop. 1

C. Calculate dual variables u and v. 1

D. "Adjust costs" replacing the costs in the non-basic positions by $c_{ij} - u_i - v_j$. 1

E. Explain why your tableau is optimal. 1/2

(vii) What is the cost associated with this optimal tableau? 1

(viii) Draw a network diagram (arrow diagram) displaying your solution. 1

(ix) Which destination(s) are under-supplied and by how much? 1

Appendix of Results

A **Algorithm 1 (Simplex Method)**

begin (Start with a Canonical tableau s.t. $\mathbf{b} \geq \mathbf{0}$.)
  while NOT finished do
    if $c_j \geq 0$ for all j
      then **STOP** (Tableau is optimal.)
      else Select j s.t. $c_j < 0$.
    fi
    if $a_{ij} \leq 0$ for all $i = 1, \ldots, m$
      then **STOP** (Problem is unbounded.)
    fi
    Select k such that:
    $\frac{b_k}{a_{kj}} = \min_i \left\{ \frac{b_i}{a_{ij}} \text{such that } a_{ij} > 0 \right\}$ (k attains the min.)
    Pivot on a_{kj}. (Divide Row k across by a_{kj} and add
  end ... multiples of Row k to the rows above & below
end ... ... introducing zeros into column j.)

B **Algorithm 2 (Dual Simplex Method)**

begin (Start with a tableau s.t. $\mathbf{c} \geq \mathbf{0}$.)
  while NOT finished do
    if $b_i \geq 0$ for all i
      then **STOP** (Tableau is optimal.)
      else Select i s.t. $b_i < 0$.
    fi
    if $-a_{ij} \leq 0$ for all $j = 1, \ldots, n$
      then **STOP** (Dual unbounded $\equiv$ Primal infeasible.)
    fi
    Select k such that:
    $\frac{c_k}{a_{ik}} = \max_j \left\{ \frac{c_j}{a_{ij}} \text{such that } a_{ij} < 0 \right\}$ (k attains max.)
    Pivot on a_{ik}. (Divide Row k across by a_{ik} and add
  end ... multiples of Row k to the rows above & below
end ... ... introducing zeros into column i.)

C The following pair of LP's are the **standard dual pair**:

$$\begin{array}{rlrl} \min & \mathbf{c}^T\mathbf{x} & \max & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b} & \text{subject to} & A^T\mathbf{y} \leq \mathbf{c} \\ & \mathbf{x} \geq 0 & & \mathbf{y} \geq 0. \end{array}$$

D Transportation Algorithm

(a) Find an initial basic feasible point $\mathbf{x}$.

(b)
- For the current solution $\mathbf{x}$ and the current cost coefficients c_{ij}, find a dual vector $(\mathbf{u}, \mathbf{v})$ such that $u_i + v_j = c_{ij}$ for all basic positions (i, j).
- Calculate the adjusted cost coefficients (a.c.c.) for all positions (i, j).

(c)
- If each a.c.c is ≥ 0, STOP.
- ELSE Pick the position with the most negative a.c.c and find the unique loop starting there (with all other positions basic).

(d)
- Shift as much as possible around the loop to get a new basic feasible point $\mathbf{x}$.
- GOTO Step 2 with the a.c.c's as the new costs.

E Notation for the Transportation Problem written as a LP in matrix notation.

- I_n is the $n \times n$ identity matrix.
- $e_n^T = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$ (n ones).
- $z_n^T = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}$ (n zeros).
- A is the $(m+n) \times (mn)$ matrix $A = \begin{bmatrix} e_n^T & z_n^T & z_n^T & \dots & z_n^T & z_n^T \\ z_n^T & e_n^T & z_n^T & \dots & z_n^T & z_n^T \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ z_n^T & z_n^T & z_n^T & \dots & e_n^T & z_n^T \\ z_n^T & z_n^T & z_n^T & \dots & z_n^T & e_n^T \\ I_n & I_n & I_n & \dots & I_n & I_n \end{bmatrix}$.
- $\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.
- The cost coefficients $\mathbf{c}$ are just the column vector of length mn:

$$\mathbf{c} = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1n} \\ c_{21} \\ c_{22} \\ \vdots \\ c_{2n} \\ \vdots \\ c_{m1} \\ c_{m2} \\ \vdots \\ c_{mn} \end{bmatrix}.$$

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