UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Department of Mathematics & Statistics
Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4303

SEMESTER: Spring 2015

MODULE TITLE: Operations Research 1

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

GRADING SCHEME:

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES:

- **Answer four of the five questions correctly for full marks** 70%**.**
- **For convenience, each question is assigned 25 marks.**
- **Ruled paper for recording transportation tableaux is appended to this Examination paper.**
- **Your examination paper must be returned with your script.**

1 (a) Oola Oil Limerick Associated (OOLA), manage an oil refining plant in Oola, Co. Limerick. The outputs from the plant are petrol and jet fuel. The refining plant consists of three separate units:

- the distillation unit (D), which produces a "feedstock" (an intermediate product)
- the cracker unit (C), which produces "fuelstock" using some of the feedstock from the distillation stage
- the blender unit (B) that blends (combines) fuelstock from the cracking unit with feedstock from the distillation unit to produce petrol & jet fuel.

Process details:

- All crude oil that arrives at D is converted into feedstock — without any wastage.
- Some of the feedstock is piped directly to B, the remainder goes to C (no waste).
- All **fuelstock** that arrives at B from C is converted into the same volume of either petrol or jet fuel or some of one & some of the other (no waste).
- All **feedstock** that arrives at B (from D) is converted into the same volume of either petrol or jet fuel or some of one & some of the other — again without waste.
- Both petrol & jet fuel can be produced from either feedstock (from the distillation unit D) or fuelstock (from the cracker unit C) during the blending process, though at different costs.

Process data:

- The plant can process 600,000 barrels of crude oil per day.
- The profit per barrel of petrol produced is €7.70 (if blended from feedstock) or €5.20 (if blended from fuelstock).
- The corresponding profit values per barrel for jet fuel are €12.30 and €10.40.
- It takes 5 barrels of crude oil to produce 1 barrel of feedstock (in the distillation unit D).
- The cracker unit C cannot process more than 40,000 barrels of feedstock per day.
- The demand for petrol & jet fuel is limited to 80,000 and 50,000 barrels per day respectively.

You should draw a simple sketch of the flow through the plant.

N.B. All feedstock not used in the cracker unit C is piped to the blender unit B to produce a combination of petrol and jet fuel as detailed above.

Formulate this problem as a LP making sure to specify decision variables, objective function, **all** the constraints and whether the problem is a max or a min one. 15

(b) Limerick Bus Services (LBS) want to re-organise their schedules so as to meet demand while minimising the number of buses required. Based on survey data, the Company estimate that the number of buses required to operate in each successive 4–hour period with the first period starting at midnight is as presented in Table 1.

Time Slot	00:00–04:00	04:00–08:00	08:00–12:00
Buses needed	4	8	10

Time Slot	12:00–16:00	16:00–20:00	20:00–00:00
Buses needed	7	12	6

Table 1: Bus Requirements

For staffing and maintenance reasons, each bus can only operate for a single block of 8 consecutive hours. However there are 6 possible starting times for each 8-hour shift, namely 00:00, 04:00, 08:00, 12:00, 16:00, and 20:00.

Formulate this problem as a LP making sure to specify decision variables, objective function, **all** the constraints and whether the problem is a max or a min one. 10

2 (a) Consider the following LP:

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ \text{subject to}\quad & x_1 + x_2 \le 4 \\ & -x_1 + 2x_2 \ge 1 \end{aligned}$$

with $x_1, x_2 \ge 0$.

(i) Reformulate the LP into a Standard Form (SF) LP. 1

(ii) Write a Simplex Tableau for your SF LP. 2

(iii) Explain why the tableau is not in Canonical Form (CF). 1/2

(iv) Use the Dual Simplex method (DSM) to pivot to CF. 4

(v) Explain why the tableau in not in Optimal Form (OF). 1/2

(vi) Perform one Simplex Method (SM) pivot. 4

(vii) Explain why the resulting tableau is not in OF and explain clearly which element of the tableau should be pivoted on using SM. **Do not perform the pivot.** 1

(viii) The tableau will be in OF after one more pivot. Perform the arithmetic necessary (**not the full pivot**) to determine the optimal values of x_1, x_2, x_3 and x_4. 4

(b) Consider the following Simplex Tableau T:

$$T = \begin{array}{|c|cccccc|} \hline & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline 11 & 0 & b & e & 0 & 0 & 0 \\ \hline a & 1 & c & 1 & 0 & 0 & 1 \\ 2 & 0 & d & -1 & 0 & 1 & 0 \\ 7 & 0 & -1 & 2 & 0 & 0 & 1 \\ \hline \end{array}$$

State conditions that the parameters a, b, c, d and e must satisfy so that

(i) T is in OF. 1

(ii) T is in unbounded form. 1

(iii) T is in one of the two infeasible forms. 1

(c) As part of the process of formulating an LP in SF, "free variables" must be expressed in terms of non-negative variables. Starting with a LP that is otherwise in SF, explain carefully how free variables can be eliminated using the expression $\mathbf{x} = \mathbf{y} - w\mathbf{e}$ where $\mathbf{e}$ is a constant vector of ones, $\mathbf{x}$ is a vector of free variables, $\mathbf{y}$ is a non-negative vector and w is a non-negative scalar variable. (For simplicity, assume that **all** the variables in the problem are free.) 5

3 (a) Start with the standard dual pair in Appendix C and show (using steps (i)–(v) below) that the optimal solutions to both are equal:

(i) Introduce slacks **s** into the min problem and show that the corresponding Simplex Tableau takes the form

$$P = \begin{array}{|c|c|c|}\hline & \mathbf{x} & \mathbf{s} \\ \hline 0 & \mathbf{c}^T & \mathbf{0}^T \\ -\mathbf{b} & -A & \mathbf{I} \\ \hline \end{array}.$$

3

(ii) Show that if P is a simplex tableau and if P^* is the OF tableau, say

$$P^* = \begin{array}{|c|c|c|}\hline & \mathbf{x} & \mathbf{s} \\ \hline -d & \mathbf{u}^T & \mathbf{v}^T \\ \mathbf{b}^* & D & B \\ \hline \end{array}$$

with optimal vector x^* and $d = \mathbf{c}^T x^*$ then there is a matrix Q (the pivot matrix) such that $P^* = QP$ where $Q = \begin{bmatrix} 1 & \mathbf{v}^T \\ \mathbf{0} & B \end{bmatrix}$ where $\begin{bmatrix} \mathbf{v}^T \\ B \end{bmatrix}$ are the right-hand m columns of P^*. 6

(iii) Now use the fact that $P^* = QP$ to show that $\mathbf{v} \geq \mathbf{0}$, $A^T\mathbf{v} \leq \mathbf{c}$ and $d = \mathbf{v}^T\mathbf{b}$. 5

(iv) Show (without using tableaux) for any primal feasible **x** and any dual feasible **y** that $\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y}$. 2

(v) Finally explain why you can conclude that $c^T x^* = b^T \mathbf{v}$ and that **v** is the optimal vector for the dual problem. 3

(b) Show that the dual of an LP in SF is 6

$$\begin{aligned} \max \quad & \mathbf{b}^T\mathbf{y} \qquad (1) \\ \text{subject to} \quad & A^T\mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \quad \text{free.} \end{aligned}$$

4 (a) Consider the optimal form (OF) tableau

$T =$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
−35	0	0	3	7	2	0	0
15	1	0	−2	5	−1	0	0
5	0	0	2	−3	3	0	1
18	0	1	−1	3	1	0	0
30	0	0	−1	−2	0	1	0

(i) Find the optimal vector and objective value when the additional requirement $x_4 = 2$ is added to the problem. 2

(ii) If the condition $x_4 = 5$ must satisfied, find the optimal vector and objective value (note that only the top two rows of the tableau and the column corresponding to the non-basic variable to be increased need to be updated). 7

(b) The following tableau T is the initial canonical form tableau for a resource allocation problem of the form max $\mathbf{c}^T\mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ with $\mathbf{x} \geq 0$:

$T =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
0	−9	−3	−1	−12	0	0	0
30	1	2	1	1	1	0	0
20	2	2	1	1	0	1	0
70	1	2	1	5	0	0	1

with optimal form (OF) tableau

$T^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
190	0	23/3	13/3	0	0	11/3	5/3
40/3	0	8/9	4/9	0	1	−4/9	−1/9
10/3	1	8/9	4/9	0	0	5/9	−1/9
40/3	0	2/9	1/9	1	0	−1/9	2/9

(i) Write the matrix Q such that $T^* = QT$. (See Q.3(a)(ii) above.) 2

(ii) Explain using Q what the effect on T^* is of adding a (positive or negative) to the inital resources available for resource 3. 2

(iii) Find the new optimal vector and objective if only 61 units of resource 3 are available instead of the original 70. 4

(c) Suppose, for a resource allocation problem as in part (b), that the price in the starting canonical form of a variable x_i that is basic in the optimal tableau is changed by the addition of q (positive or negative) to the price.

(i) Show that the effect on the optimal tableau is to add q times the row in T^* associated with x_i to the top (objective) row of T^*. 4

(ii) Suppose that the price associated with variable x_1 in the tableau T in part (b) is increased from €9 to €10. What is the effect on the objective function z? 1

(iii) What is the maximum amount by which the price associated with variable x_1 may be **decreased** while keeping the currently basic variables basic? 3

5 (a) The (balanced) Transportation Problem

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} \qquad \sum_{j=1}^{n} x_{ij} = a_i, \text{for } i = 1, \ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \text{for } j = 1, \ldots, n.$$

$$x_{ij} \geq 0, \text{for each } i \text{ and } j.$$

with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} = b_j$ can be written as an LP in Standard Form:

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

referring to Appendix E for the definitions of A, $\mathbf{b}$ and $\mathbf{c}$. (Note that $\mathbf{b}$ is **not** the vector of demands $b_1, \ldots, b_n$.)

(i) Use the expression for the dual of an LP in SF in Eq. 1 in Q3 (b) above to show that the dual of the Transportation Problem is 7

$$\max \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

subject to

$u_i + v_j \leq c_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$

$\mathbf{u}$&$\mathbf{v}$ free.

(ii) Explain briefly, without further algebra but referring to the result (established in Q.3 (a)) that the optimal objective values for a primal-dual pair are equal, how this result is used to determine whether a transportation tableau is optimal. 2

(b) A company has three manufacturing plants, all making the same product. Plants 1, 2 and 3 produce 15, 10 & 5 items respectively per day. There are four outlets that require the product. Unfortunately, some of the plants can only ship to certain destinations.

- Plant 1 can ship to all four outlets with unit shipping costs of 2, 5, 4 and 3 respectively.
- Plant 2 can only ship to outlets 1, 2 & 4 with unit shipping costs of 7, 2 and 6 respectively.
- Plant 3 can ship to all four outlets with unit shipping costs of 1, 4, 3 & 6 respectively.

The four outlets require 10, 15, 20 and 15 items per day respectively. There is a shortfall in production capacity.

(i) Formulate this problem as a transportation problem by writing a transportation tableau. (Note that supply does not equal demand so a dummy supply with zero shipping costs is needed. Assign a "large" positive cost N to the forbidden link .) 2

(ii) A. Use the SCEM method to find an initial basic feasible solution. **N.B. Make the choice of first basic position that eliminates the last row and the column containing** N**.** 2

B. Now calculate dual variables u and v. 2

C. Adjust costs replacing costs in non-basic positions by $c_{ij} - u_i - v_j$. 2

D. Is the tableau optimal? Explain why or why not. 1

(iii) If the NWCM is used to generate a starting tableau, several iterations of the Transportation algorithm are needed. At one of these iterations the following tableau is generated. (Refer to the statement of the Transportation Algorithm in App. D if you wish.)

0^{10}	6	1	0^5
2	0^{15}	$(N-6)$	0^5
-1	5	0^5	3
1	4	0^{15}	0^5

A. Find the "loop" connecting a succession of basic positions, starting at the non-basic position with most negative cost. 2

B. Make the max allowable increase/decrease in x_{ij} for the basic positions in the loop. 1

C. Calculate dual variables u and v. 2

D. Adjust costs replacing costs in the non-basic positions by $c_{ij} - u_i - v_j$.

(iv) Draw a network diagram (arrow diagram) displaying your solution. 1

(v) Which destination(s) are under-supplied and by how much? 1

Appendix of Results

A Algorithm 1 (Simplex Method)

begin (Start with a Canonical tableau s.t. $\mathbf{b} \geq \mathbf{0}$.)
  while NOT finished do
    if $c_j \geq 0$ for all j
      then STOP (Tableau is optimal.)
      else Select j s.t. $c_j < 0$.
    fi
    if $a_{ij} \leq 0$ for all $i = 1, \dots, m$
      then STOP (Problem is unbounded.)
    fi
    Select k such that:
    $\frac{b_k}{a_{kj}} = \min_i \left\{ \frac{b_i}{a_{ij}} \text{such that } a_{ij} > 0 \right\}$ (k attains the min.)
    Pivot on a_{kj}. (Divide Row k across by a_{kj} and add
  end … multiples of Row k to the rows above & below
end … … … introducing zeros into column j.)

B Algorithm 2 (Dual Simplex Method)

begin (Start with a tableau s.t. $\mathbf{c} \geq \mathbf{0}$.)
  while NOT finished do
    if $b_i \geq 0$ for all i
      then STOP (Tableau is optimal.)
      else Select i s.t. $b_i < 0$.
    fi
    if $-a_{ij} \leq 0$ for all $j = 1, \dots, n$
      then STOP (Dual unbounded $\equiv$ Primal infeasible.)
    fi
    Select k such that:
    $\frac{c_k}{a_{ik}} = \max_j \left\{ \frac{c_j}{a_{ij}} \text{such that } a_{ij} < 0 \right\}$ (k attains max.)
    Pivot on a_{ik}. (Divide Row k across by a_{ik} and add
  end … multiples of Row k to the rows above & below
end … … … introducing zeros into column i.)

C The following pair of LP's are the **standard dual pair**:

$$\begin{array}{rlcrl} \min & \mathbf{c}^T\mathbf{x} & \qquad & \max & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b} & & \text{subject to} & A^T\mathbf{y} \leq \mathbf{c} \\ & \mathbf{x} \geq 0 & & & \mathbf{y} \geq 0. \end{array}$$

D Transportation Algorithm

(a) Find an initial basic feasible point $\mathbf{x}$.

(b)
- For the current solution $\mathbf{x}$ and the current cost coefficients c_{ij}, find a dual vector $(\mathbf{u}, \mathbf{v})$ such that $u_i + v_j = c_{ij}$ for all basic positions (i, j).
- Calculate the adjusted cost coefficients (a.c.c.) for all positions (i, j).

(c)
- If each a.c.c is ≥ 0, STOP.
- ELSE Pick the position with the most negative a.c.c and find the unique loop starting there (with all other positions basic).

(d)
- Shift as much as possible around the loop to get a new basic feasible point $\mathbf{x}$.
- GOTO Step 2 with the a.c.c's as the new costs.

E Notation for the Transportation Problem written as a LP in matrix notation.

- I_n is the $n \times n$ identity matrix.
- $e_n^T = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$ (n ones).
- $z_n^T = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}$ (n zeros).
- A is the $(m+n) \times (mn)$ matrix $A = \begin{bmatrix} e_n^T & z_n^T & z_n^T & \dots & z_n^T & z_n^T \\ z_n^T & e_n^T & z_n^T & \dots & z_n^T & z_n^T \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ z_n^T & z_n^T & z_n^T & \dots & e_n^T & z_n^T \\ z_n^T & z_n^T & z_n^T & \dots & z_n^T & e_n^T \\ I_n & I_n & I_n & \dots & I_n & I_n \end{bmatrix}$.
- $\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.
- The cost coefficients $\mathbf{c}$ are just the column vector of length mn:

$$\mathbf{c} = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1n} \\ c_{21} \\ c_{22} \\ \vdots \\ c_{2n} \\ \vdots \\ c_{m1} \\ c_{m2} \\ \vdots \\ c_{mn} \end{bmatrix}.$$

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