UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4303

SEMESTER: Spring 2014

MODULE TITLE: Operations Research 1

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella/Dr. M. Devine

GRADING SCHEME:

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES:

- **Answer four of the five questions correctly for full marks $83\frac{1}{3}\%$.**
- **Use separate Answer Books for Part 1 (Q1–Q3) and Part 2 (Q4–Q5).**
- **Your project work will be graded out of the remaining $16\frac{2}{3}\%$.**
- **For convenience, each question is assigned 25 marks.**
- **Ruled paper for recording transportation tableaux is appended to this Examination paper.**
- **Your examination paper <u>must</u> be returned with your script.**

1 A Bakery makes four products, Bread, Cake, Muffins & Scones. The production process uses three main ingredients; Flour, Oil & Yeast.

- The table shows the amount of each ingredient needed to make one unit of each of the four products.

	Amount Needed To Make				Amount
Ingredient	Bread	Cake	Muffins	Scones	Available
Flour	1	1	0	2	12
Oil	2	2	2	1	18
Yeast	1	5	1	1/2	5

- The unit sales price for Bread, Cake, Muffins & Scones are €6, €5, €7 and €3 respectively.
- The company wants to maximise total sales income.

(a) Formulate the problem as a Standard Form (S.F.) LP. 2

(b) Write a Simplex Tableau for this LP. 2

(c) Is the tableau in Canonical Form? Explain briefly. 1

(d) Perform one iteration of the Simplex Method — explain why you selected the pivot row & column. (Note that only 2 of the 4 rows change.) 7

(e) Is the tableau in Optimal Form? Why/why not? 1

(f) What is the optimal production plan and optimal revenue? 1

(g) For strategic reasons, the company needs to ensure that the combined number of units of Bread & Scones produced is at least 6. Add this constraint to your LP — keeping it in S.F. 1

(h) Modify your optimal tableau to include this extra constraint. 1

(i) Is the tableau in Canonical Form? Explain briefly. 1

(j) Pivot once using the Dual Simplex Method to pivot the problem into Canonical Form. (Note that 5 of the 9 columns do not change.) 7

(k) Interpret the resulting tableau. What does it tell you about the LP? 1

2 The following tableau P is the initial canonical form tableau for a resource allocation problem of the form

max $\mathbf{c}^T\mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ with $\mathbf{x} \geq 0$:

$P =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
0	−7	−11	−2	−6	0	0	0
20	2	1	1	1	1	0	0
10	1	0	−2	1	0	1	0
5	1	1	−1	0	0	0	1

The optimal tableau is:

$P^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
$152\frac{1}{2}$	$10\frac{1}{2}$	0	0	$\frac{1}{2}$	$6\frac{1}{2}$	0	$4\frac{1}{2}$
$7\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$
25	2	0	0	2	1	1	−1
$12\frac{1}{2}$	$1\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$

(a) What is the minimum amount that I should charge for 10 units of resource 1 so that the total revenue from selling these 10 units and from selling products is at least €$152\frac{1}{2}$? 2

(b) What is the minimum amount that I should charge for 20 units of resource 1 so that the total revenue from selling these 20 units and from selling products is at least €$152\frac{1}{2}$? 4

N.B. See Appendix D for the tableaux that result from pivoting on P^* in various positions.

(c) What is the new optimal vector x^* if I need to make 12 units of product 4? 2

(d) What is the new optimal vector x^* if I need to make 14 units of product 4? (**The Tableau resulting from pivoting on the x_4 column of the optimal tableau in a particular row is given in Appendix E. If you use this tableau explain carefully why.**) 4

(e) What is the new optimal vector x^* if only 2 units of resource 3 are available instead of the original 5? 3

(f) Find a matrix Q such that $P^* = QP$. 2

(g) What is the maximum amount by which the price of product 1 may be increased without changing the optimal vector? 3

(h) Give the range within which the selling price of product 2 may change without changing the optimal vector. 5

3 A company has four manufacturing plants, all making the same product. Plants 1, 2, 3 and 4 produce 20, 5, 10 & 20 items respectively per day. There are three outlets that require the product. Unfortunately, some of the plants can only ship to certain destinations.

- Plant 1 can ship to all three outlets with unit shipping costs of 6, 7 and 2 respectively.
- Plant 2 can only ship to outlets 1 & 2 with unit shipping costs of 3 and 8 respectively.
- Plant 3 can ship to all three outlets with unit shipping costs of 2, 9 & 4 respectively.
- Plant 4 can only ship to outlets 1 & 3 with unit shipping costs of 6 & 1 respectively.

The three outlets require 10, 20 and 15 items per day respectively. Due to the excess of production capacity, over-production occurs.

(a) Formulate this problem as a transportation problem by writing a transportation tableau. (Note that supply does not equal demand so a dummy demand with zero shipping costs is needed. Assign a "large" positive cost N to the forbidden links.) 2

(b) Use the NW Corner method to find an initial basic feasible solution. 2

(c) Use the SCEM method to find an initial basic feasible solution. 2

(d) Find the costs of these two starting solutions. Which solution has the lower cost? 1

(e) **Starting with the SCEM solution, <u>not</u> the NWCM solution**, perform **two** iterations of the Transportation Algorithm: 13

- calculate dual variables u and v
- adjust costs replacing costs in non-basic positions by $c_{ij} - u_i - v_j$
- find the "loop" connecting a succession of basic positions, starting at the non-basic position with most negative cost
- make max allowable increase in x_{ij} for the basic positions in the loop.

(f) Carefully note the reduction in cost achieved at each of the two iterations. 1

(g) Explain carefully why, after two iterations, the transportation tableau is optimal. 1

(h) Check (using the original costs) that the cost of your final (optimal) tableau is equal to the cost of the SCEM tableau reduced by the total reduction in cost achieved over the two iterations. 1

(i) Draw a network diagram (arrow diagram) illustrating your solution. 1

(j) In which manufacturing plants are each day's surplus production kept? 1

4 (a) Suppose a farmer can choose between one of three actions: use his land for grazing, use his land to plant corn, or use his land to grow potatoes. Nature takes one of following three states: heavy rainfall, moderate rainfall, light rainfall. The gains of the farmer according to the action taken and the level of rainfall are given below:

	Heavy rain	Moderate rain	Light rain
Grazing	5	15	5
Corn	-10	25	15
Potatoes	-20	20	30

The probabilities of heavy, moderate and light rainfall are as 0.25, 0.5 and 0.25, respectively. Show which action, with workings, should the farmer take according to:

(i) the expected value criterion 2

(ii) the Laplace criterion 2

(iii) the minimax criterion 3

(iv) the Savage (regret) criterion 5

(v) the Hurwitz criterion with coefficient of optimism $\alpha = 0.8$ 2

(b) Suppose the utility of the farmer from his gains is $u(x) = \sqrt{20 + x}$. Show which action, with workings, should they take in order to maximise his expected utility? 3

(c) Given the weather that actually occurred, the following were found to be the probabilities of the three possible forecasts by the weather expert:

- given that heavy rain occurred, the forecast made was heavy rain with probability 0.6, moderate rain with probability 0.3 and light rain with probability 0.1.
- given that moderate rain occurred, the forecast made was heavy rain with probability 0.2, moderate rain with probability 0.6 and light rain with probability 0.2.
- given that light rain occurred, the forecast made was that there will be heavy rain with probability 0.1, moderate rain with probability 0.3 and light rain with probability 0.6.

(i) Calculate the posterior probabilities for the level of rainfall given that the forecast of the expert is for heavy rain. 7

(ii) Given that the farmer maximises his expected reward, derive the optimal decision given that the expert forecasts heavy rain. 1

5 (a) The state of the economy in a particular year can be described by a random variable Y which has a normal distribution with mean $\mu = 3$ and standard deviation $\sigma = 2$. An individual can choose one of 3 possible investments:

- Government bonds: Always gives a payoff of $X_1 = 11$.
- Low risk stock: Gives payoff $X_2 = 9 + Y$
- High risk stock: Gives payoff $X_3 = 7 + 3Y$

Suppose the utility of the investor from her payoff is $u(x) = x^2$. Show, with workings, which investment should the individual take in order to maximise their expected utility? 8

(b) The following table describes the tasks involved in a project (in days) and their prerequisites.

Task	Duration	Prerequisites
A	5	-
B	7	-
C	10	A
D	6	A,B
E	8	C
F	7	E
G	12	D,E
H	4	F

(i) Draw a network representing the process. 5

(ii) Using this network, estimate the expected minimum completion time and the corresponding critical path. 7

(iii) Give the time window for the initiation of each task. 5

Appendix of Results

A **Algorithm 1 (Simplex Method)**

begin (Start with a Canonical tableau s.t. $\mathbf{b} \geq \mathbf{0}$.)
  while NOT finished do

```
            if  c_j ≥ 0 for all j
              then STOP (Tableau is optimal.)
              else Select j s.t. c_j < 0.
            fi
            if a_ij ≤ 0 for all i = 1, ..., m
              then STOP (Problem is unbounded.)
            fi
            Select k such that:
            b_k/a_kj = min_i { b_i/a_ij such that a_ij > 0 } (k attains the min.)
            Pivot on a_kj. (Divide Row k across by a_kj and add
    end     ... multiples of Row k to the rows above & below
end         ...   ...   ... introducing zeros into column j.)
```

B **Algorithm 2 (Dual Simplex Method)**

```
begin (Start with a tableau s.t. c ≥ 0.)
    while NOT finished do
            if  b_i ≥ 0 for all i
              then STOP (Tableau is optimal.)
              else Select i s.t. b_i < 0.
            fi
            if −a_ij ≤ 0 for all j = 1, ..., n
              then STOP (Dual unbounded ≡ Primal infeasible.)
            fi
            Select k such that:
            c_k/a_ik = max_j { c_j/a_ij such that a_ij < 0 } (k attains max.)
            Pivot on a_ik. (Divide Row k across by a_ik and add
    end     ... multiples of Row k to the rows above & below
end         ...   ...   ... introducing zeros into column i.)
```

C The following pair of LP's are the **standard dual pair**:

$$\begin{array}{llcll} \min & \mathbf{c}^T\mathbf{x} & \qquad & \max & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b} & & \text{subject to} & A^T\mathbf{y} \leq \mathbf{c} \\ & \mathbf{x} \geq 0 & & & \mathbf{y} \geq 0. \end{array}$$

D The results of pivoting on the highlighted elements in the s_1 column in the optimal tableau P^* for Q.2(a) are shown:

$P^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
$152\frac{1}{2}$	$1\frac{1}{2}$	0	0	$\frac{1}{2}$	$6\frac{1}{2}$	0	$4\frac{1}{2}$
$7\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	**0.5**	0	$-\frac{1}{2}$
25	2	0	0	2	1	1	-1
$12\frac{1}{2}$	$1\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$

$\longrightarrow$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
55	4	0	-13	-6	0	0	11
15	1	0	2	1	1	0	-1
10	1	0	-2	1	0	1	0
5	1	1	-1	0	0	0	1

$P^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
$152\frac{1}{2}$	$1\frac{1}{2}$	0	0	$\frac{1}{2}$	$6\frac{1}{2}$	0	$4\frac{1}{2}$
$7\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0.5	0	$-\frac{1}{2}$
25	2	0	0	2	**1**	1	-1
$12\frac{1}{2}$	$1\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$

$\longrightarrow$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
-10	$-2\frac{1}{2}$	0	0	$-12\frac{1}{2}$	0	$-6\frac{1}{2}$	11
-5	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0
25	2	0	0	2	1	1	-1
0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1

$P^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
$152\frac{1}{2}$	$1\frac{1}{2}$	0	0	$\frac{1}{2}$	$6\frac{1}{2}$	0	$4\frac{1}{2}$
$7\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0.5	0	$-\frac{1}{2}$
25	2	0	0	2	1	1	-1
$12\frac{1}{2}$	$1\frac{1}{2}$	1	0	$\frac{1}{2}$	**0.5**	0	$\frac{1}{2}$

$\longrightarrow$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
-10	-9	-13	0	-6	0	0	-2
-5	-1	-1	1	0	0	0	-1
0	-1	-2	0	1	0	1	-2
25	3	2	0	1	1	0	1

E The results of pivoting on the highlighted element in the x_4 column in the optimal tableau P^* for Q.2(a) is shown:

$P^* =$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
$152\frac{1}{2}$	$1\frac{1}{2}$	0	0	$\frac{1}{2}$	$6\frac{1}{2}$	0	$4\frac{1}{2}$
$7\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0.5	0	$-\frac{1}{2}$
25	2	0	0	**2**	1	1	-1
$12\frac{1}{2}$	$1\frac{1}{2}$	1	0	$\frac{1}{2}$	0.5	0	$\frac{1}{2}$

$\longrightarrow$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
$146\frac{1}{4}$	10	0	0	0	$6\frac{1}{4}$	$-\frac{1}{4}$	$4\frac{3}{4}$
$1\frac{1}{4}$	0	0	1	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
$12\frac{1}{2}$	1	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$6\frac{1}{4}$	1	1	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$