UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4303

SEMESTER: Spring 2013

MODULE TITLE: Operations Research 1

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella/Dr. M. Devine

GRADING SCHEME:

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES:

- **Answer four of the five questions correctly for full marks $83\frac{1}{3}\%$.**
- **Use separate Answer Books for Part 1 (Q1–Q3) and Part 2 (Q4–Q5).**
- **Your project work will be graded out of the remaining $16\frac{2}{3}\%$.**
- **For convenience, each question is assigned 25 marks.**
- **You may bring a single handwritten (one-sided) A4 sheet of paper into the exam with useful material.**
- **An Appendix contains some relevant results and a table of the standard Normal distribution.**
- **You will be provided with ruled paper for recording transportation tableaux.**

1 A company uses two machines to make two different products.

- For a single production run, the following table shows the hours on each machine needed to make one unit of each product:

Machine	Time Used To Make Product 1	Time Used To Make Product 2	Total Machine Time Available in Production Run
1	1	2.0	10
2	3	4.5	20

- The unit sales price for Products 1 & 2 are €2 and €3 respectively.
- The company wants to maximise total sales income.

(a) Formulate the problem as a Standard Form (S.F.) LP. 2

(b) Write a Simplex Tableau for this LP. 2

(c) Is the tableau in Canonical Form? Explain briefly. 1

(d) Perform one iteration of the Simplex Method. 7

(e) Is the tableau in Optimal Form? Why/why not? 1

(f) Read off the solution from the tableau. 1

(g) For strategic reasons, the company needs to ensure that $2x_1 + x_2 \geq 15$ where x_1 and x_2 are the number of units of Products 1 and 2 produced. Add this constraint to your LP — keeping it in S.F. 1

(h) Write a Simplex Tableau for this modified LP. 2

(i) Pivot **once** using the Subproblem Method to pivot the problem into Canonical Form. 7

(j) Interpret the resulting tableau. What does it tell you about the LP? 1

2 The following LP represents a company's income maximisation problem, subject to resource constraints:

$$\max z = 3x_1 + 2x_2 + 5x_3$$
$$\text{subject to}$$
$$x_1 + 2x_2 + x_3 \le 43 \quad \text{Resource 1}$$
$$3x_1 + 0x_2 + 2x_3 \le 46 \quad \text{Resource 2}$$
$$x_1 + 4x_2 + 0x_3 \le 42 \quad \text{Resource 3}$$
$$x_1, x_2, x_3 \ge 0,$$

its optimal tableau is

$T_1 =$

	x_1	x_2	x_3	s_1	s_2	s_3
135	4	0	0	1	2	0
10	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0
23	$1\frac{1}{2}$	0	1	0	$\frac{1}{2}$	0
2	2	0	0	-2	1	1

(a) What is the minimum price that the company should charge for three units of resource 1 if the total revenue is to remain ≥ 135? 1

(b) What is the maximum number of units of resource 1 that I can sell while keeping x_2 and x_3 (and s_3) basic? Explain briefly. 1

(c) Does decreasing the availibility of resource 3 by one unit change

(i) the solution?

(ii) the optimal value of z?

Explain briefly. 1+1

(d) Show that the pair of LP's **P** and **D** :

$$\min c^T x \quad (\mathbf{P}) \qquad\qquad \max\ -b^T y \quad (\mathbf{D})$$
$$\text{subject to } Ax \le b, \qquad \text{subject to } -A^T x \le c,$$
$$x \ge 0. \qquad\qquad y \ge 0.$$

are a dual pair (based on the standard dual pair in App. C). 3

(e) By introducing non-negative "slack" vectors **u** and **s** to the primal & dual problem respectively, represent **P** and **D** as Standard Form Simplex tableaux. 2

Q.2 continued on next page.

(f) Suppose that $\mathbf{c} \geq 0$ but $\mathbf{b}$ has one or more negative elements. 1+1+2+2

(i) Is **P** in Canonical Form? Explain briefly.

(ii) Is **D** in Canonical Form? Explain briefly.

(iii) How may **D** be pivoted to optimality using the Simplex Method?

(iv) Explain carefully how the pivot rules for **D** may be translated into pivot rules for **P** — in other words, the Dual Simplex Method.

(g) I need to add the constraint $x_2 \leq 9$ to the problem. Using the ruled paper provided (if you wish), amend the optimal tableau given at the beginning of this Question to include this extra constraint. 2

(h) Your new tableau is not in Canonical Form — which row & column would you pivot on to ensure that all the columns of the 5×5 identity matrix appear in the tableau with zeros in the coefficient row? (Do not perform the pivot.) 1

(i) The amended tableau takes the form

$T_2 =$

	x_1	x_2	x_3	s_1	s_2	s_3	s_4
135	4	0	0	1	2	0	0
10	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	0
23	$1\frac{1}{2}$	0	1	0	$\frac{1}{2}$	0	0
2	2	0	0	-2	1	1	0
-1	$\frac{1}{4}$	0	0	$-\frac{1}{2}$	$\frac{1}{4}$	0	1

Is it in Canonical Form? Explain briefly. 1

(j) If not, which row & column would you select using the Dual Simplex pivot rule? 1

(k) What would be the updated z–value after the pivot? (Do not perform the full pivot, just the small amount of arithmetic necessary to answer this question.) 3

(l) Why has the optimal z–value decreased from 135? Explain briefly. 2

3 A company has three manufacturing plants, all making the same product. Plants 1,2 and 3 can produce 10, 20 and 20 items per day. There are four outlets that require the product. Unfortunately, some of the plants can only ship to certain destinations.

- Plant 1 can ship to all four outlets with unit shipping costs of 4, 3, 2 and 5 respectively.
- Plant 2 can only ship to outlets 1, 2 and 3 with unit shipping costs of 2, 1 and 4 respectively.
- Plant 3 can only ship to outlets 2, 3 and 4 with unit shipping costs of 4, 2 and 7 respectively.

The four outlets require 10, 25, 10 and 15 items per day respectively. Due to the shortage of production capacity, local outlet managers must purchase stock locally as necessary.

(a) Formulate this problem as a transportation problem by writing a transportation tableau. (Note that supply does not equal demand so a dummy producer with zero shipping costs is needed. Assign a "large" positive cost N to the forbidden links.) 2

(b) Use the NW Corner method to find an initial basic feasible solution. 2

(c) Use the SCEM method to find an initial basic feasible solution. 2

(d) Find the costs of these two starting solutions. Which solution has the lower cost? 2

(e) Starting with the NWCM solution, perform **one** iteration of the Transportation Algorithm

- calculate dual variables u and v 2
- adjust costs replacing costs in non-basic positions by $c_{ij} - u_i - v_j$ 2
- find the "loop" connecting a succession of basic positions, starting at the non-basic position with most negative cost 1
- make max allowable increase in x_{ij} for the basic positions in the loop. 1

(f) What is the reduction in cost achieved in this iteration? 1

(g) The following transportation tableau (for the same problem) was obtained by starting with the SCEM solution and applying several iterations of the Transportation Algorithm. The adjusted costs have just been updated.

-1	-1	0	0^{10}
0^{5}	0^{15}	5	$N-2$
$N-5$	0^{10}	0^{10}	2
0^{5}	1	3	0^{5}

- Find the "loop" connecting a succession of basic positions. 1
- Make the max allowable increase in x_{ij} for the basic positions in the loop. 1
- Calculate the dual variables and update the adjusted costs. 2+2

(h) Is the tableau now optimal? Explain why/why not. 1

(i) Draw a network diagram (arrow diagram) illustrating your solution. 2

(j) What local purchases must be made in which outlets each day to make up for the shortfall in supply? 1

4 (a) Suppose a farmer can choose between one of three actions: use his land for grazing, use his land to plant corn, or use his land to grow potatoes. Nature takes one of following three states: heavy rainfall, moderate rainfall, light rainfall. The gains of the farmer according to the action taken and the level of rainfall are given below:

	Heavy rain	Moderate rain	Light rain
Grazing	5	10	10
Corn	-5	30	15
Potatoes	-15	20	25

The probabilities of heavy, moderate and light rainfall are as 0.45, 0.3 and 0.25, respectively. Which action should the farmer take according to:

(i) the expected value criterion 2

(ii) the Laplace criterion 2

(iii) the minimax criterion 3

(iv) the Savage (regret) criterion 5

Q.4 continued on next page.

(b) The state of the economy in a particular year can be described by a random variable Y which has a uniform distribution on $[0, 1]$. An individual can choose one of 3 possible investments:

- Government bonds: Always gives a payoff of $X_1 = 1.01$.
- Low risk stock: Gives payoff $X_2 = 0.9 + 0.2Y$
- High risk stock: Gives payoff $X_3 = 0.7 + 0.6Y$

(i) Which action should the investor take according to the Hurwitz criterion with coefficient of optimism $\alpha = 0.4$ 4

(ii) Suppose the utility of the investor from their payoff is $u(x) = x^2$. Which investment should the individual take in order to maximise their expected utility? 9

5 A process is defined in the following table, where a denotes the optimistic duration, m the most likely duration and b the pessimistic duration of a task. The expected value for each task is given by $E[.] = \frac{a+4m+b}{6}$ while the variance is given by $Var[.] = \frac{(b-a)^2}{36}$

Task	Prerequisites	a	m	b
A	-	4	7	16
B	-	3	5	7
C	A	2	6	10
D	A, B	4	6	8
E	C	3	7	11
F	D	5	7	15
G	D, E	2	5	8

(a) Draw a network representing the process. 5

(b) Using this network, estimate the expected minimum completion time and the corresponding critical path. 7

(c) Give the time window for the initiation of each task. 5

(d) Estimate the distribution of the completion time for the critical path. You may assume this distribution is normal. 2

(e) Estimate the probability that the critical path takes more than 30 time units. 3

(f) Discuss why the assumption of normality may not be suitable. 3

Appendix of Results

A **Algorithm 1 (Simplex Method)**

```
begin (Start with a Canonical tableau s.t. b ≥ 0.)
  while NOT finished do
      if c_j ≥ 0 for all j
        then STOP (Tableau is optimal.)
        else Select j s.t. c_j < 0.
      fi
      if a_ij ≤ 0 for all i = 1, ..., m
        then STOP (Problem is unbounded.)
      fi
      Select k such that:
      b_k/a_kj = min_i { b_i/a_ij such that a_ij > 0 } (k attains the min.)
      Pivot on a_kj. (Divide Row k across by a_kj and add
  end ... multiples of Row k to the rows above & below
end   ...  ...  ... introducing zeros into column j.)
```

B **Algorithm 2 (Dual Simplex Method)**

```
begin (Start with a tableau s.t. c ≥ 0.)
  while NOT finished do
      if b_i ≥ 0 for all i
        then STOP (Tableau is optimal.)
        else Select i s.t. b_i < 0.
      fi
      if −a_ij ≤ 0 for all j = 1, ..., n
        then STOP (Dual unbounded ≡ Primal infeasible.)
      fi
      Select k such that:
      c_k/a_ik = max_j { c_j/a_ij such that a_ij < 0 } (k attains max.)
      Pivot on a_ik. (Divide Row k across by a_ik and add
  end ... multiples of Row k to the rows above & below
end   ...  ...  ... introducing zeros into column i.)
```

C The following pair of LP's are the **standard dual pair**:

$$\begin{array}{rlcrl} \min & \mathbf{c}^T\mathbf{x} & \qquad & \max & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b} & & \text{subject to} & A^T\mathbf{y} \leq \mathbf{c} \\ & \mathbf{x} \geq 0 & & & \mathbf{y} \geq 0. \end{array}$$

Tail Probabilities for the Standard Normal Distribution

This gives the probability that $P(Z > a)$.

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
3.6	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000