# Ch.4 Exercises ①

1 (a) ✓

(b) $x_2$ <u>non basic</u> so LH col $\to \begin{bmatrix} -75 - 4x_2 \\ \hline 10 + x_2 \\ 5 - x_2 \\ 20 + 2x_2 \\ 30 - x_2 \end{bmatrix}$ still feas if $x_2 \leq 5$ (min row rat.)

when $x_2 = 3$, LH col $\to \begin{bmatrix} -87 \\ \hline 13 \\ 2 \\ 26 \\ 27 \end{bmatrix}$ so $x \to \begin{bmatrix} 13 \\ 3 \\ 26 \\ 0 \\ 0 \\ 27 \\ 2 \end{bmatrix}$

(c) No, as min row ratio is 5 (LP is infeas. if this constraint added.)
(If I pivot $x_2$ up to 5, the row defining $x_2$ has no neg elements in T, so the non-basic $x_2$ cannot be increased further.)

~~(d)~~.

1(d). $5 = x_2 + 2x_4 + 3x_5 + x_7$. All coeffs $\geq 0$ so each term $\leq 5$

$\Rightarrow x_2 \leq 5,\ x_4 \leq \frac{5}{2},\ x_5 \leq 5/3,\ x_7 \leq 5$.

(e). $\underline{x_1 \text{ is basic}} = 10$. Want an increase of 2 in $x_1$.

write $x_1 = 10 + \overset{(4)}{x_2} + \overset{(3)}{2x_4} + \overset{(2)}{x_5}$ $\leftarrow$ obj coeffs. So, inc $x_2$ by 2,

or $x_4$ by 1 or $x_5$ by 2. Choose the (nonbasic) variable to inc. which gives smallest net increase in $z$. Examine the obj coefficients, $\Delta x_2 = 2 \Rightarrow \Delta z = 8$, $\boxed{\Delta x_4 = 1 \Rightarrow \Delta z = 3}$, $\Delta x_5 = 2 \Rightarrow \Delta z = 4$. So increase $x_4$ from $0 \to 1$.

Note, $1 <$ min row ratio for $x_4$ ie 5.

so LHcol $\to$

$$\begin{bmatrix} \underline{-75 \ -3x_4} \\ 10 + 2x_4 \\ 5 - 2x_4 \\ 20 + 1x_4 \\ 30 + 1x_4 \end{bmatrix} = \begin{bmatrix} \underline{-78} \\ 12 \\ 3 \\ 21 \\ 31 \end{bmatrix} \text{ and } x \to \begin{bmatrix} 12 \\ 0 \\ 21 \\ 1 \\ 0 \\ 31 \\ 3 \end{bmatrix}$$

(f)

As $x_1$ is basic, $x_1 = 10$, inc. $x_4$ up to its min row ratio of 2.5 by pivoting on. $R_2$, $x_4$ col ( ~~R~~ Pivot(T, 3, 5))

$$
\begin{array}{c}
\begin{array}{cccccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7
\end{array} \\
\left[\begin{array}{c|ccccccc}
-82\frac{1}{2} & 0 & 2\frac{1}{2} & 0 & 0 & -2\frac{1}{2} & 0 & -1\frac{1}{2} \\
\hline
15 & 1 & 0 & 0 & 0 & 2 & 0 & 1 \\
2.5 & 0 & 0 & 0 & 1 & 1\frac{1}{2} & 0 & \frac{1}{2} \\
22\frac{1}{2} & 0 & -1\frac{1}{2} & 1 & 0 & 2\frac{1}{2} & 0 & \frac{1}{2} \\
32\frac{1}{2} & 0 & 1\frac{1}{2} & 0 & 0 & 1\frac{1}{2} & 1 & \frac{1}{2}
\end{array}\right]
\end{array}
\rightarrow x = \begin{bmatrix} 15 \\ 0 \\ 22\frac{1}{2} \\ 2\frac{1}{2} \\ 0 \\ 32\frac{1}{2} \\ 0 \end{bmatrix}
$$

So cannot inc. $x_1$ further as $x_1$ row is

(*) $[15 \mid 1 \; 0 \; 0 \; 0 \; 2 \; 0 \; 1]$ ($x_1$ now $= 15$)

So $x_1 = 21$ is infeasible.

In fact I only need to add $R_2$ to $R_1$ in opt Tableau to get (*) - new $R_1$.

No need to calculate a full pivot!

3 B

(+).

As $x_1$ is basic ($x_1 = 10$) Inc $x_4$ up to max value (2.5)

LH Col.

$$\begin{bmatrix} -75 - 3x_4 \\ \hline 10 + 2x_4 \\ 5 - 2x_4 \\ 20 + 1x_4 \\ 30 + 1x_4 \end{bmatrix} \rightarrow \begin{bmatrix} -82\tfrac{1}{2} \\ \hline 15 \\ 0 \\ 22\tfrac{1}{2} \\ 32\tfrac{1}{2} \end{bmatrix} \quad \text{so } x \rightarrow \begin{bmatrix} 15 \\ 0 \\ 22\tfrac{1}{2} \\ 2\tfrac{1}{2} \\ 0 \\ 32\tfrac{1}{2} \\ 0 \end{bmatrix}$$

consistant with result of pivoting.

~~Either~~ Any inc in $x_4$ beyond $\frac{5}{2}$ would make $x_2 < 0$
$\rightarrow$ infeasible.

2(a) Opt dual soln $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

(b) $Q = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}$

(c) 8 * sh price for $R_2$ $= 8 * €3 = €24$.

(d) Min row ratio for $s_2$ is 10. Cost is $\underline{3 \times 10 = 30}$.

So pivot on $R_2$, $s_2$ col to inc $s_2$ to 10 first $\to$

$$\left[\begin{array}{c|cccccccc} 1\cancel{2}0 & -3 & 0 & 1 & \overset{x_4}{\boxed{13}} & 0 & 0 & 4 \\ \hline 20 & 0 & 0 & 0 & -4 & 1 & 0 & -1 \\ 10 & 1 & 0 & 0 & -4 & 0 & 1 & -1 \\ 15 & 0 & 1 & \frac{1}{2} & 2\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right]$$

Now solve for $s_2$ from $R_2$

$s_2 = 10 + 4x_4$ ($x_4$ non basic)

Inc $x_4$ by 3 units so $s_4 \uparrow 10 + 12 = 22$

extra cost is $13 * 3 = 39$

So total cost is $\underline{€30 + €39} = €69$.

($z$ $\downarrow$ to 81).

4A

2. (e) $x_3$ is non-basic. Min row ratio is 20 so can inc $x_3$ $\uparrow 20$ without ch. in basis.

So LH col $\begin{bmatrix} \frac{150 - x_3}{20} \\ 10 \\ 10 - \frac{1}{2}x_3 \end{bmatrix} \xrightarrow{x_3=0} \begin{bmatrix} \frac{140}{20} \\ 10 \\ 5 \end{bmatrix} \therefore x = \begin{bmatrix} 10 \\ 5 \\ 10 \\ 0 \end{bmatrix}$

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2.(f). Max row ratio for $x_3$ is 20 so first pivot on $R_3$ + $x_3$ col to ~~get~~ inc $x_3 \uparrow 20$.

Now the row defining $x_3$ is $[20 \mid 0 \; 2 \; 1 \; 9 \; 0 \; -1 \; 1]$

So I can write ~~$x_1 = 20 - 2x_2 + x_3 - 9x_4$~~

$x_3 = 20 + s_2$.

Inc. $s_2$ from 0 (as non-basic) to 5. (less the min row ratio of 10).

New tab is

| | | | | | | | $s_2$ | |
|---|---|---|---|---|---|---|---|---|
| $-4s_2+$ | 130 | 0 | -2 | 0 | -8 | 0 | 4 | -1 |
| | 20 | 0 | 0 | 0 | -4 | 1 | 0 | -1 |
| $-s_2+$ | 10 | 1 | 0 | 0 | -4 | 0 | 1 | -1 |
| $+s_2+$ | 20 | 0 | 2 | 1 | 9 | 0 | -1 | +2 |

Only need to calc indicated row + cols in tab

So $s_2 = 5$ ~~th~~ changes LH col to $\begin{bmatrix} 110 \\ \hline 20 \\ 5 \\ 25 \end{bmatrix}$ and $x \to \begin{bmatrix} 5 \\ 0 \\ 25 \\ 0 \end{bmatrix}$

6

2 (g) From scratch: Lower left (4,1) el of starting tab $\to$ 30 - 5 $\therefore$ $P^* \to P^* + Q \begin{bmatrix} \phi & \\ -5 & 0 \end{bmatrix}$

$$= P^* + \begin{bmatrix} -5\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} & 0 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} 150 & -5 & \\ \hline 20 & +5 & \\ 10 & +5 & \text{unchanged} \\ 10 & -5 & \end{array}\right]$$

So $x \to \begin{bmatrix} 15 \\ 5 \\ 0 \\ 0 \end{bmatrix}$ $\left( s \to \begin{bmatrix} 25 \\ 0 \\ 0 \end{bmatrix} \right)$

$z \to 145$

(h) replace -5 by +5

So $x \to \begin{bmatrix} 5 \\ 15 \\ 0 \\ 0 \end{bmatrix}$, $s \to \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix}$

$z \to 155$

2 (i) Product 1 (A) sp = 7.

$x_1$ basic in opt soln.

From scratch, price $\to 7+\xi$.

$\to \quad p^* \to p^* + \left[\begin{array}{c|cc} & -\xi & \\ \hline \phi & \phi & \phi \end{array}\right]$

Pivot as usual so $p^*$ in opt form.

Top row $\to \left[150+10\xi \mid 0, 0, 1, 1-4\xi, 0, 3+\xi, 1-\xi\right]$

so opt provided $-3 \le \xi \le 1/4$

so sp for $x_1$ in range €4 — €7¼.

3(a) $Q = \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Cost: $5 * €5 = 25$

(b) Min ratio for $s_2$ is 5 so first pivot up to 5. ($R_2$, $s_2$ col)

new $R_2$ : ~~[5 | 0 − 1 1 1~~

$[5 \mid 0 \;\; -1 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; -1]$

so $s_2 = 5 + x_2 + s_3$ so inc $x_2$ or $s_3$ by 3

new top row is $[30 \mid 0 \;\; \overset{}{\underset{*}{5}} \;\; 2 \;\; -5 \;\; 0 \;\; 0 \;\; \underset{*}{6}]$

$x_2$ min row ratio in new tab is 5

$s_3$ " " " " " " is 5.

Cheaper to inc $x_2$ as obj coeff is 5 vs 6 for $s_3$

So $\Delta$ Cost $= 3 * €5 = €15$

Total $\Delta$ Cost $= €25 + €15 = \underline{€40}$.

3(c). From scratch, $P \to$

$$P + \begin{bmatrix} 0 & \\ 0 & \\ 0 & O \\ -1 & \end{bmatrix} ; \quad QZ = -1 \begin{bmatrix} 1 & \\ -1 & \\ -1 & \phi \\ 1 & \end{bmatrix}$$

$z \uparrow$

$\cancel{\&}$ So LH col of $P^* \to \begin{bmatrix} 55-1 \\ 5+1 \\ 5+1 \\ 5-1 \end{bmatrix} = \begin{bmatrix} 54 \\ 6 \\ 6 \\ 4 \end{bmatrix}$; $x \to \begin{bmatrix} 4 \\ 0 \\ 0 \\ 6 \end{bmatrix}$

$z \to 54$.

3(d). $QZ = +3 \begin{bmatrix} 1 & \\ -1 & \\ -1 & \phi \\ 1 & \end{bmatrix}$ so LH Col of $P^* \to$

$$\begin{bmatrix} 55+3 \\ 5-3 \\ 5-3 \\ 5+3 \end{bmatrix} = \begin{bmatrix} 58 \\ 2 \\ 2 \\ 8 \end{bmatrix} \quad x \to \begin{bmatrix} 8 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$z \to 58$.

(e) $P_1$ sales price changes from €6 $\to$ €6 + $\xi$

$x_1$ basic in opt tab

as change to min.

So (from scratch) $P_0 \to P_0 + \begin{bmatrix} \phi & -\xi & \phi & 0 \end{bmatrix}$

and $P^* \to P^* + \begin{bmatrix} \phi & -\xi & \phi & 0 \end{bmatrix}$

So top row in $P^*$ is now $[\,55 \mid -\xi \;\; 0 \;\; 7 \;\; 0 \;\; 0 \;\; 5 \;\; 1\,]$

Add $\xi R_3$ to top row to eliminate $-\xi$ at top of $x_1$ col. (Tableau now in opt. form).

top row $\to [\,55 + 5\xi \mid 0 \;\; \xi \;\; 7+\xi \;\; 0 \;\; 0 \;\; 5 \;\; 1+\xi\,]$

So $\underline{\xi \geq 0}$. no <u>decrease</u> in sales price poss without changing opt voc.

(f).

(7). Product 3 ($x_3$) non basic in opt tab.

change sales price from $€\cancel{7}^4 \to €\cancel{7}^4 + r$.

$\sim\!\triangleright$ following usual procedure.

Opt tab top row $\to [55 \mid 0 \quad 0 \quad \underline{7-r} \quad 0 \quad 0 \quad 5 \quad 1]$

So $r = 7 \equiv$ sales price of $x_3 \to €11$.

ensures obj coeff of $x_3$ to 0 ($x_3$ becomes basic)

(8). Product 4 – $x_4$ has sales price of €5.

~~Bas~~ Basic in opt tableau.

$€5 \to €5 + \varepsilon$

Follow standard procedure:

top row in $P^* \to [55 \mid 0 \quad 0 \quad 7 \quad -\varepsilon \quad 0 \quad 5 \quad 1]$

Pivot to elim. $-\varepsilon$

Add. $\varepsilon * R_2$ to top row

Top Row in $P^* \to [55+5\varepsilon \mid 0, -\varepsilon, 7+\varepsilon, 0, 0, 5+\varepsilon, 1-\varepsilon]$

(5) Cont. So $-5 \leq \xi \leq 0$ leaves soln unchanged

So $\xi = 1$ $\Rightarrow$ Top Row $\rightarrow [60 \mid 0, -1, 8, 0, 0, 6, 0]$

(Salesprice inc from €5 $\rightarrow$ €6)

Have neg coeff in top ~~col~~ row, <u>$x_2$ col</u>.

<u>Pivot</u> on $R_3$, $x_2$ col: Add $R_3$ to top Row ($R_0$) and Row 2

$$\text{Tab} \rightarrow \left[\begin{array}{c|ccccccc} 60+5 & 0+1, & 0, & 8+1, & 0, & 0, & 6, & 0+1 \\ \hline 5 & 0 & 0, & -2, & 0, & 1 & -1 & -1 \\ 10 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\ \hline 5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

<u>unch</u> $\rightarrow$

So ~~xa~~ $x = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 10 \end{bmatrix}$; $z \rightarrow 65$.

4 (a). 50 & 40 in $R_1$, LH col of $P_0$

so $P_0 \to P_0 + \left[\begin{array}{c|c} 0 & \\ \hline -10 & 0 \\ 0 & \\ 0 & \end{array}\right]$

and $P^* \to P^* - 10 \begin{bmatrix} 3 & \\ 1 & \\ -1 & \phi \\ 0 & \end{bmatrix}$

ie $P^* \to \left[\begin{array}{c|c} 380-30 & * \\ \hline 40-10 & \\ 30+10 & * \\ 10 & \end{array}\right]$ so $z \to 350$

$x \to \begin{bmatrix} 30 \\ 10 \\ 40 \\ 0 \end{bmatrix}$

(b) (extra value of malt ($R_1$) $\to$ 50+15).

$P^\Delta \to P^* + 15 \begin{bmatrix} 3 & \\ +1 & \\ -1 & \phi \\ 0 & \end{bmatrix}$; $P^* \to \begin{bmatrix} 380+45 \\ 40+15 \\ 30-15 \\ 10 \end{bmatrix} = \begin{bmatrix} 425 \\ 55 \\ 15 \\ 10 \end{bmatrix}$

so $x \to \begin{bmatrix} 55 \\ 10 \\ 15 \\ 0 \end{bmatrix}$

4 (c). $\Delta = -35$

LH col of $P^* \to \begin{bmatrix} 380 \\ \hline 40 \\ 30 \\ 10 \end{bmatrix} - 45 \begin{bmatrix} 3 \\ \hline 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 245 \\ \hline -5 \\ 75 \\ 10 \end{bmatrix}$

Infeas (non-canonical) so use $\underline{DSM}$ on $R_2$, US1 col

either $x_4$ col (col ratio $\frac{7}{-4}$) or $S_3$ col (col ratio $\frac{1}{-2}$)

So Pick $S_3$ col.

$$P^* \to \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccccc} & x_1 & \overbrace{x_2 \quad x_3}^{\text{Basic}} & & & & & \text{Basic} \end{array}$$

$$\left[\begin{array}{c|ccccccc} 242\frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 & 3\frac{1}{2} & 1\frac{1}{2} & 0 \\ \hline \frac{5}{2} & -\frac{1}{2} & 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{2} & 1 \\ \hline 72\frac{1}{2} & \frac{1}{2} & 0 & 1 & -1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 5 & 1 & 1 & 0 & 3 & 1 & 0 & 0 \end{array}\right]$$

No need to calc full tab, just LH. col + $x_2, x_3$ cols.

So $x = \begin{bmatrix} 0 \\ 5 \\ 72\frac{1}{2} \\ 0 \end{bmatrix}$.