UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science & Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4105

SEMESTER: Autumn 2016

MODULE TITLE: Linear Algebra 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

PERCENTAGE OF TOTAL MARKS: 80%

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks.**

1 (a) Show that if A is an $m \times n$ matrix (real or complex) then (Rank-Nullity Thm.)

$$\dim \text{nullsp}(A) + \text{rank}(A) = n.$$

Hint: Let $U_k = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a basis for the nullspace of A (i.e. $\dim \text{nullsp}\, A = k$) and extend the set U_k to $B_n = \{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_{n-k}\}$ to form a basis for $\mathbb{R}^n$. Then show that $\{A\mathbf{v}_1, \dots A\mathbf{v}_{n-k}\}$ form a basis for the range of A. 8

(b) Show that if a square matrix A is of full rank, then there exists a matrix B such that $AB = I$. Hint: use the fact that a set of n linearly independent vectors in an n–dimensional vector space (such as $\mathbb{R}^n$ or $\mathbb{C}^n$) must form a basis for the space. 4

(c) Show that if for a square matrix A there is a matrix B s.t. $AB = I$, then

(i) $BA = I$. Hint: use the result of part (b) of this question and the Rank–Nullity Thm. (part (a) of this question). 6

(ii) B is unique. 2

(d) (i) Show that for compatible matrices A and B that $(AB)^* = B^* A^*$ where A^* is the complex conjugate of A transpose. 4

(ii) Show that if A and B are $n \times n$ invertible matrices then $(AB)^{-1} = B^{-1}A^{-1}$. 1

2 (a) Show that for any $\mathbf{x} \in \mathbb{R}^n$,

(i) $\|\mathbf{x}\|_1 \le \sqrt{n}\|\mathbf{x}\|_2$ 6
and

(ii) $\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$. 2

(b) Given a choice of norms $\|\cdot\|$ on $\mathbb{C}^m$ and $\mathbb{C}^n$, the induced matrix norm of the $m \times n$ matrix A is defined as

$$\|A\| \equiv \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| \equiv \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$

(i) Explain carefully why the two conditions 2+2

$$\|A\mathbf{x}\| \le B, \text{ for all unit vectors } \mathbf{x},$$
$$\|A\mathbf{x}_0\| = B, \text{ for some unit vector } \mathbf{x}_0$$

imply that $\|A\| = B$.

(ii) Show using the definition of an induced matrix norm and the results in part (a) of this question that for any $m \times n$ matrix A,

$$\|A\|_1 \le \sqrt{m}\|A\|_2.$$

4

(c) Show that for any $m \times n$ matrix A, the eigenvalues of A^*A are real and non-negative. 2

(d) Show that for any $m \times n$ matrix A, the eigenvectors of A^*A form an orthonormal basis for $\mathbb{C}^n$. Hint: assume that A has a Singular Value Decomposition (SVD) $A = U\Sigma V^*$ where U is $m \times m$ unitary, V is $n \times n$ unitary and Σ is an $m \times n$ diagonal matrix of singular values. 2

(e) Using the two conditions in part (b)(i) of this question and the result in part (d), show that the 2–norm $\|A\|_2$ of a $m \times n$ matrix A can be calculated as the square root of the largest eigenvalue of A^*A:

$$\|A\|_2 = \sqrt{\lambda_{\text{max}}(A^*A)}$$

5

3 (a) Show that for every $m \times n$ complex matrix A we can write: 15

$$A_1 \equiv U^*AV = \begin{bmatrix} \sigma & 0 \\ 0 & B \end{bmatrix} \tag{1}$$

where $\sigma = \|A\|$, B is an $(m-1) \times (n-1)$ matrix, $U = \begin{bmatrix} y_0 & U_1 \end{bmatrix}$ and $V = \begin{bmatrix} x_0 & V_1 \end{bmatrix}$, the unit vector x_0 satisfies $\|Ax_0\| = \|A\|$ and finally $y_0 = \frac{Ax_0}{\|A\|}$. The matrices U_1 and V_1 are chosen so that U and V are unitary $m \times m$ and $n \times n$ respectively.

(The vector and matrix 2–norm are used throughout this question. You may assume that $\|OA\| = \|A\|$ for any unitary matrix O.)

(b) Using part (a) prove by induction on the size of A that every $m \times n$ complex matrix A has a Singular Value Decomposition (SVD) $A = U\Sigma V^*$ where U is $m \times m$ unitary, V is $n \times n$ unitary and Σ is an $m \times n$ diagonal matrix of singular values. 10

4 Recall that a complex $m \times m$ matrix P is a projection operator if $P^2 = P$.

(a) Show that if λ is an eigenvalue of a projection operator P then either $\lambda = 0$ or $\lambda = 1$. 1

(b) Given an $m \times n$ complex matrix A with $m \geq n$, let P be the orthogonal projection operator onto the range of A. Show that for any $\mathbf{v} \in \mathbb{C}^m$, $(I - P)\mathbf{v}$ is orthogonal to the range of A. 2

(c) Given an $m \times n$ complex matrix A with $m \geq n$, show that A^*A is invertible if and only if A has full rank (remember that a square matrix is invertible if and only if $Ax = 0$ implies $x = 0$.) 3+3

(d) Given a linearly independent set of vectors $\{a_1, \dots, a_n\}$ in $\mathbb{C}^m$, let A be the $m \times n$ matrix whose j^{th} column is a_j. Explain why A^*A is invertible and use the result in (b) to show that P, the orthogonal projection operator onto the range of A, is given by the formula 6

$$P = A(A^*A)^{-1}A^*.$$

(e) Use the results in parts (b) and (d) to show that the vector $x \in \mathbb{R}^n$ that minimises $\|Ax - b\|_2^2$ is just the vector x satisfying $Pb = Ax$ where P is the projection operator defined in (c) and referred to in (d). Hint: first show that $y = Pb$ is the vector in the range of A closest to b. 5

(f) Given an $n \times m$ matrix A with QR factorisation $A = QR$ and reduced QR factorisation $A = \hat{Q}\hat{R}$, show that the vector x that minimises $\|Ax - b\|_2^2$ is the solution to $\hat{R}x = \hat{Q}^*b$. 4

(g) Comment briefly on the significance of the latter result. 1

5 (a) For any vector $v \in \mathbb{C}^k$, let the matrix $H = I - 2P_v$ (where $P_v = \dfrac{vv^*}{v^*v}$).

(i) Show with a sketch that the effect on an arbitrary vector $x \in \mathbb{C}^k$ of left-multiplying x by H is to reflect x in $P_{\perp v}x$, the normal to v in the x–v plane. 2

(ii) Find the choices of vector v that make Hx, the **Householder reflection** of x, return a multiple of e_1 where $e_1 \in \mathbb{C}^k$ is a vector of zeroes with one in the first position. 12

(iii) Which of the two choices found should be used and why? 2

(b) The following algorithm (Alg. 0.1) takes as its input an arbitrary $m \times n$ complex matrix A. Explain the effect of line 5 and relate it to the Householder reflection in part (a). 3

Algorithm 0.1 *Householder QR Factorisation*

```
(1) for k = 1 to n
(2)     x = A_{k:m,k}
(3)     v_k = x + sign(x_1)||x||e_1
(4)     v_k = v_k/||v_k||
(5)     A_{k:m,k:n} = A_{k:m,k:n} - 2v_k (v_k^* A_{k:m,k:n})
(6) end
```

(c) The work done in Alg. 0.1 is dominated by the (implicit) inner loop `j=k:n` over the columns of the submatrix $A_{k:m,k:n}$ in line 5. Show that the total operation count W for the algorithm is $W = 2mn^2 - 2/3n^3$ to leading order. 6

(Just show that the coefficient of n^3 in W is $-2/3$ and that the coefficient of mn^2 is 2.)

6 For any $m \times m$ matrix A, Gauss Elimination without pivoting consists of:

Algorithm 0.2 *Gauss Elimination Without Pivoting — in words*

```
for k = 1 to m − 1
    Add suitable multiples of row k to the rows beneath
    to introduce zeroes below the main diagonal in column k.
end
```

(a) Show that each iteration of the above algorithm can be effected by left-multiplying A by a matrix $L_k = I - \ell_k e_k^*$ where ℓ_k is the vector of **multipliers** for the k^{th} column of A (the first k entries of ℓ_k are 0) and e_k is a vector in $\mathbb{C}^n$ with one in the k^{th} position and zeroes elsewhere. Give a simple formula for the non-zero entries of ℓ_k. 5

(b) Show that for each k, $L_k^{-1} = I + \ell_k e_k^*$. 2

(c) Show that the matrix $L = L_1{}^{-1} L_2{}^{-1} \dots L_{m-1}{}^{-1}$ is just $I + \ell_1 e_1^* + \cdots + \ell_m e_m^*$. 2

(d) Explain briefly why the result in (c) means that L is lower triangular. 1

(e) What special structure does A have after the algorithm has completed? 1
(See over for the rest of Q.6.)

(f) When partial pivoting is applied, we have

$$L_{m-1}P_{m-1}L_{m-2}P_{m-2}\dots L_2P_2L_1P_1A = U$$

where each P_j swaps row j with one of the rows $j+1, \dots, m$ (if necessary) to make the absolute value of the "pivot" A_{jj} as large as possible. Defining

$$\begin{aligned}\Pi_j &= P_{m-1}P_{m-2}\dots P_j \\ L'_j &= \Pi_{j+1}L_j\Pi_{j+1}^{-1},\end{aligned}$$

show that 6

$$L_{m-1}P_{m-1}L_{m-2}P_{m-2}\dots L_2P_2L_1P_1 = L'_{m-1}L'_{m-2}\dots L'_2L'_1 \quad \Pi_1.$$

(g) Show that the LU factorisation $A = LU$ (without pivoting) is now replaced by $PA = LU$ (with pivoting) where $P = \Pi_1$ and $L = L_1^{\prime -1}L_2^{\prime -1}\dots L_{m-1}^{\prime -1}$. 2

(h) Finally, show that the matrix L is lower triangular as it was in the no-pivoting case. 6