UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science & Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4105

SEMESTER: Autumn 2015

MODULE TITLE: Linear Algebra 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

PERCENTAGE OF TOTAL MARKS: 70%

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks, 70%.**

1 (a) Prove the following Lemma: Let V be a finite-dimensional vector space. Let $L = \{\underline{l}_1, \ldots, \underline{l}_n\}$ be a linearly independent set in V and let $S = \{\underline{s}_1, \ldots, \underline{s}_m\}$ be a second subset of V which spans V. Then $n \leq m$. 10

Hint: Write each of the $\underline{l}_i$ in L as a linear combination of the spanning set S and use the fact that a homogeneous linear system $A^T c = 0$ with more unknowns than equations has non-trivial solutions where A is the coefficient matrix for expressing the vectors in L in terms of the vectors in S.

(b) Show that this result leads to a definition for the dimension of a vector space. 2

(c) Let S and T be subsets of a vector space V such that $S \subseteq T$.

(i) Prove that if S is linearly dependent then so is T. 2

(ii) Prove that (i) is equivalent to the statement that if T is linearly independent, so is S. 1

(d) The Adding/Removing Theorem states that:

Theorem 0.1 *Let S be a non-empty set of vectors in a vector space V. Then:*

(i) if S is a linearly independent set and if $\mathbf{v} \in V$ is **outside** span(S) *(i.e. $\mathbf{v}$ cannot be expressed as a linear combination of the vectors in S) then the set $S \cup \mathbf{v}$ is still linearly independent (i.e. adding $\mathbf{v}$ to the list of vectors in S does not affect the linear independence of S),*

(ii) if $v \in S$ is a vector that is expressible as a linear combination of other vectors in S and if $S \setminus \mathbf{v}$ means S with the vector v removed then S and $S \setminus \mathbf{v}$ span the same space, i.e.

$$\text{span}(S) = \text{span}(S \setminus \mathbf{v}).$$

Use the Adding/Removing Theorem to prove **one of the following**: 10

Theorem 0.2 *If V is an n-dimensional vector space and if S is a set in V with exactly n elements then S is a basis for V if* **either**

(i) S spans V

(ii) or S is linearly independent.

(You may assume that all bases for a given vector space have the same number of elements.)

2 (a) Show that for any $\mathbf{x} \in \mathbb{R}^n$,

(i) $\|\mathbf{x}\|_1 \le \sqrt{n}\|\mathbf{x}\|_2$ 6

and

(ii) $\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$. 2

(b) Given a choice of norms $\|\cdot\|$ on $\mathbb{C}^m$ and $\mathbb{C}^n$, the induced matrix norm of the $m \times n$ matrix A is defined as

$$\|A\| \equiv \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| \equiv \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$

(i) Explain carefully why the two conditions 3

$$\|A\mathbf{x}\| \le B, \text{ for all unit vectors } \mathbf{x},$$
$$A\mathbf{x}_0 = B, \text{ for some unit vector } \mathbf{x}_0$$

imply that $\|A\| = B$.

(ii) Show using the definition of an induced matrix norm and the results in Q.2(a) that for any $m \times n$ matrix A,

$$\|A\|_1 \le \sqrt{n}\|A\|_2.$$

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(c) Show that for any $m \times n$ matrix A, the eigenvalues of A^*A are real and non-negative. 2

(d) Using the two conditions in part (a), show that the induced 2–norm $\|A\|_2$ of a $m \times n$ matrix A can be calculated as the square root of the largest eigenvalue of A^*A:

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)}$$

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(e) Use the result found in part (d) to find the induced 2–norm of the matrix $A = \begin{bmatrix} -1 & 3 \\ 3 & -4 \\ 1 & 7 \end{bmatrix}$.

(i) First show that the eigenvalues of A^*A are 75 and 10. (Just check that $\lambda_1 = 75$ and $\lambda_2 = 10$ are roots of the characteristic polynomial of A^*A.) 2

(ii) Why does this imply that $\|A\|_2 = \sqrt{75}$? 1

3 (a) Show that for every $m \times n$ complex matrix A we can write: 12

$$A_1 \equiv U^* A V = \begin{bmatrix} \sigma & 0 \\ 0 & B \end{bmatrix} \qquad (1)$$

where $\sigma = \|A\|$, B is an $(m-1) \times (n-1)$ matrix, $U = \begin{bmatrix} y_0 & U_1 \end{bmatrix}$ and $V = \begin{bmatrix} x_0 & V_1 \end{bmatrix}$, the unit vector x_0 satisfies $\|Ax_0\| = \|A\|$ and finally $y_0 = \frac{Ax_0}{\|A\|}$. The matrices U_1 and V_1 are chosen so that U and V are unitary $m \times m$ and $n \times n$ respectively.

(The vector and induced matrix 2–norm is used throughout this question. You may assume that $\|OA\| = \|A\|$ for any unitary matrix O.)

(b) Using part (a) prove by induction on the size of A that every $m \times n$ complex matrix A has a Singular Value Decomposition (SVD) $A = U\Sigma V^*$ where U is $m \times m$ unitary, V is $n \times n$ unitary and Σ is an $m \times n$ diagonal matrix of singular values. 6

(c) Use the SVD to show that for any $m \times n$ matrix A, the singular values σ_i (the diagonal elements of Σ) may be found by computing the eigenvalues λ_i of A^*A and taking square roots — i.e. $\sigma_i = \sqrt{\lambda_i}$ and that the unitary $n \times n$ matrix V has the eigenvectors of A^*A as its columns. 2

(d) The reduced SVD expresses A as $A = \hat{U}\hat{\Sigma}V^*$ where $\hat{U}$ consists of the first n columns of U and $\hat{\Sigma}$ the first n rows of Σ. Derive the equation $AV = \hat{U}\hat{\Sigma}$ and explain how it may be used to find the reduced matrix $\hat{U}$. 2

(e) Given the matrix

$$A = \begin{pmatrix} 0 & 7 \\ 4 & 0 \\ 0 & -5 \end{pmatrix}$$

find a reduced SVD of A using any method you wish . (Remember that the singular values are in non-increasing order: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.)

3

4 Recall that a complex $m \times m$ matrix P is a projection operator if $P^2 = P$.

(a) Show that if λ is an eigenvalue of a projection operator P then either $\lambda = 0$ or $\lambda = 1$. 1

(b) Given an $m \times n$ complex matrix A with $m \geq n$, show that A^*A is invertible if and only if A has full rank (remember that a square matrix is invertible if and only if $Ax = 0$ implies $x = 0$.) 6

(c) Given a linearly independent set of vectors $\{a_1, \ldots, a_n\}$ in $\mathbb{C}^m$, let A be the $m \times n$ matrix whose j^{th} column is a_j. Use the result in (b) to show that P, the orthogonal projection operator onto the range of A, is given by the formula 6

$$P = A(A^*A)^{-1}A^*.$$

(d) Let A be an $m \times n$ complex matrix with $m \geq n$ and let $b \in \mathbb{C}^m$ be given. For any $x \in \mathbb{R}^n$, define the residual r by $r \equiv Ax - b$. Show that the following four conditions are equivalent (show that the first implies the second, etc.) 8

$$r \perp \text{range}(A) \tag{2}$$
$$A^*r = 0 \tag{3}$$
$$A^*Ax = A^*b \tag{4}$$
$$Pb = Ax \tag{5}$$

where $P \in \mathbb{C}^{m \times m}$ is the orthogonal projection operator onto the range of A found in part (c).

(e) Use the results in (d) to show that the vector $x \in \mathbb{R}^n$ that minimises $\|Ax - b\|_2^2$ is just the vector x satisfying $Pb = Ax$ where P is the projection operator defined in (c) and referred to in (d). 4

5 (a) For any vector $v \in \mathbb{C}^k$, let the matrix $H = I - 2P_v$ (where $P_v = \dfrac{vv^*}{v^*v}$).

(i) Show with a sketch that the effect on an arbitrary vector $x \in \mathbb{C}^k$ of left-multiplying x by H is to reflect x in $P_{\perp v}x$, the normal to v in the x–v plane. 2

(ii) Find the choices of vector v that make Hx, the **Householder reflection** of x, return a multiple of e_1 where $e_1 \in \mathbb{C}^k$ is a vector of zeroes with one in the first position. 10

(iii) Which of the two choices found should be used and why? 1

(b) The following algorithm (Alg. 0.1) takes as its input an arbitrary $m \times n$ complex matrix A. Explain the effect of line 5 and relate it to the Householder reflection in part (a). 3

Algorithm 0.1 *Householder QR Factorisation*

```
(1)     for k = 1 to n
(2)         x = A_{k:m,k}
```

(3) $v_k = x + \text{sign}(x_1)\|x\|e_1$
(4) $v_k = v_k/\|v_k\|$
(5) $A_{k:m,k:n} = A_{k:m,k:n} - 2v_k\,(v_k^* A_{k:m,k:n})$
(6) end

(c) The work done in Alg. 0.1 is dominated by the (implicit) inner loop `j=k:n` over the columns of the submatrix $A_{k:m,k:n}$ in line 5. Show that the total operation count W for the algorithm is $W = 2mn^2 - 2/3n^3$ to leading order. 9

(Just show that the coefficient of n^3 in W is $-2/3$ and that the coefficient of mn^2 is 2.)

6 For any $m \times m$ matrix A, Gauss Elimination without pivoting consists of:

Algorithm 0.2 *Gauss Elimination Without Pivoting — in words*

for $k = 1$ to $m - 1$
    Add suitable multiples of row k to the rows beneath
    to introduce zeroes below the main diagonal in column k.
end

(a) Show that each iteration of the above algorithm can be effected by left-multiplying A by a matrix $L_k = I - \ell_k e_k^*$ where ℓ_k is the vector of **multipliers** for the k^{th} column of A (the first k entries of ℓ_k are 0) and e_k is a vector in $\mathbb{C}^n$ with one in the k^{th} position and zeroes elsewhere. Give a simple formula for the non-zero entries of ℓ_k. 5

(b) Show that for each k, $L_k^{-1} = I + \ell_k e_k^*$. 2

(c) Show that the matrix $L = L_1{}^{-1}L_2{}^{-1}\dots L_{m-1}{}^{-1}$ is just $I + \ell_1 e_1^* + \dots + \ell_m e_m^*$. 2

(d) Explain briefly why the result in (c) means that L is lower triangular. 1

(e) What special structure does A have after the algorithm has completed? 1
(See over for the rest of Q.6.)

(f) When partial pivoting is applied, we have

$$L_{m-1}P_{m-1}L_{m-2}P_{m-2}\dots L_2P_2L_1P_1A = U$$

where each P_j swaps row j with one of the rows $j+1, \dots, m$ (if necessary) to make the absolute value of the "pivot" A_{jj} as large as possible. Defining

$$\begin{aligned} \Pi_j &= P_{m-1}P_{m-2}\dots P_j \\ L_j' &= \Pi_{j+1}L_j\Pi_{j+1}^{-1}, \end{aligned}$$

show that 6

$$L_{m-1}P_{m-1}L_{m-2}P_{m-2}\dots L_2P_2L_1P_1 = L_{m-1}'L_{m-2}'\dots L_2'L_1' \quad \Pi_1.$$

(g) Show that the LU factorisation $A = LU$ (without pivoting) is now replaced by $PA = LU$ (with pivoting) where $P = \Pi_1$ and $L = L_1'^{-1}L_2'^{-1}\dots L_{m-1}'^{-1}$. 2

(h) Finally, show that the matrix L is lower triangular as it was in the no-pivoting case. 6