UNIVERSITY *of* LIMERICK
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Faculty of Science & Engineering
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END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4105

SEMESTER: Autumn 2014

MODULE TITLE: Linear Algebra 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

PERCENTAGE OF TOTAL MARKS: 70%

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks, 70%.**

1 (a) The Adding/Removing Theorem states that:

Theorem 0.1 *Let S be a non-empty set of vectors in a vector space V. Then:* 10

(i) if S is a linearly independent set and if $\mathbf{v} \in V$ is **outside** $\mathrm{span}(S)$ *(i.e. $\mathbf{v}$ cannot be expressed as a linear combination of the vectors in S) then the set $S \cup \mathbf{v}$ is still linearly independent (i.e. adding $\mathbf{v}$ to the list of vectors in S does not affect the linear independence of S),*

(ii) if $v \in S$ is a vector that is expressible as a linear combination of other vectors in S and if $S \setminus \mathbf{v}$ means S with the vector v removed then S and $S \setminus \mathbf{v}$ span the same space, i.e.

$$\mathrm{span}(S) = \mathrm{span}(S \setminus \mathbf{v}).$$

Prove only one of (i) & (ii).

(b) Use the Adding/Removing Theorem to prove that: 10

Theorem 0.2 *If V is an n-dimensional vector space and if S is a set in V with exactly n elements then S is a basis for V if* **either**

(i) S spans V

(ii) or S is linearly independent.

(You may assume that all bases for a given vector space have the same number of elements.)

Prove only one of the two results.

(c) Prove the Cauchy-Schwarz Inequality: if V is a real Inner Product Space, then for any $\underline{u}, \underline{v} \in V$, $\left|\left\langle \underline{u}, \underline{v} \right\rangle\right| \leq \|\underline{u}\| \|\underline{v}\|$. 4

(d) Define $\cos\theta$, where θ is the angle between two vectors $\underline{u}, \underline{v}$ in a real Inner Product Space V. Use the Cauchy-Schwarz Inequality to show that $|\cos\theta| \leq 1$. 1

2 Given a choice of norm $\|\cdot\|$ on $\mathbb{C}^m$ and $\mathbb{C}^n$, the induced matrix norm of the $m \times n$ matrix A is defined as

$$\|A\| \equiv \sup_{\|x\|=1} \|Ax\|.$$

(a) Explain carefully why the two conditions 5

$$\|Ax\| \leq B, \text{for all unit vectors } x,$$
$$\|Ax_0\| = B, \text{for some unit vector } x_0$$

imply that $\|A\| = B$.

(b) Using the two conditions in part (a), show that the induced 1–norm $\|A\|_1$ of an $m \times n$ matrix A can be calculated as the "maximum column sum" of A:

$$\|A\|_1 = \max_{1 \leq i \leq n} \|a_i\|_1.$$

where a_i stands for the i^{th} column of A. 10

(c) Use the result found in part (b) to calculate the induced 1–norm of the 3×2 matrix $A = \begin{bmatrix} 4-i & 3+2i \\ 7-i & i \\ 2-3i & 1+4i \end{bmatrix}$. 3

(**No need to evaluate the square roots.**)

(d) Again using the two conditions in part (a), show for any rank 1 $m \times n$ matrix $A = uv^*$ (where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$) that the induced 2–norm of A is given by

$$\|A\|_2 = \|u\|_2 \|v\|_2.$$

(Hint; check that the unit vector $x_0 = v/\|v\|$ attains the bound.) 6

(e) Find the induced 2–norm of the matrix uv^* where $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$. 1

3 Recall that a complex $m \times m$ matrix P is a projection operator if $P^2 = P$.

(a) Show that if λ is an eigenvalue of a projection operator P then either $\lambda = 0$ or $\lambda = 1$. 1

(b) Given an $m \times n$ complex matrix A with $m \geq n$, show that A^*A is invertible if and only if A has full rank (remember that a square matrix is invertible if and only if $Ax = 0$ implies $x = 0$.) 6

(c) Given a linearly independent set of vectors $\{a_1, \ldots, a_n\}$ in $\mathbb{C}^m$, let A be the $m \times n$ matrix whose j^{th} column is a_j. Use the result in (b) to show that P, the orthogonal projection operator onto the range of A, is given by the formula 8

$$P = A(A^*A)^{-1}A^*.$$

(d) Let A be an $m \times n$ complex matrix with $m \geq n$ and let $b \in \mathbb{C}^m$ be given. For any $x \in \mathbb{R}^n$, define the residual r by $r \equiv Ax - b$. Show that the following four conditions are equivalent (show that the first implies the second, etc.) 10

$$r \perp \text{range}(A) \qquad (1)$$
$$A^*r = 0 \qquad (2)$$
$$A^*Ax = A^*b \qquad (3)$$
$$Pb = Ax \qquad (4)$$

where $P \in \mathbb{C}^{m \times m}$ is the orthogonal projection operator onto the range of A found in part (c).

4 (a) For any vector $v \in \mathbb{C}^k$, let the matrix $H = I - 2P_v$, where $P_v = \dfrac{vv^*}{v^*v}$. Show with a sketch that the effect on an arbitrary vector $x \in \mathbb{C}^k$ of left-multiplying x by H is to reflect x in $P_{\perp v}x$, the normal to v in the x–v plane. 2

(b) Find the choices of vector v that make Hx, the **Householder reflection** of x, return a multiple of e_1 where $e_1 \in \mathbb{C}^k$ is a vector of zeroes with one in the first position. (Full marks will be awarded if the case where x is complex is correctly analysed.) 12

(c) Explain briefly how the following algorithm (Alg. 0.1) (taking as its input an arbitrary $m \times n$ complex matrix A) implements the choice of v found in part (b). 4

Algorithm 0.1 *Householder QR Factorisation*

for $k = 1$ to n
 $x = A_{k:m,k}$
 $v_k = x + \text{sign}(x_1)\|x\|e_1$
 $v_k = v_k/\|v_k\|$
 $A_{k:m,k:n} = A_{k:m,k:n} - 2v_k\,(v_k^* A_{k:m,k:n})$
end

(d) The work done in Alg. 0.1 is dominated by the (implicit) inner loop `j=k:n` over the columns of the submatrix $A_{k:m,k:n}$ in line 5. Show that the total operation count W for the algorithm is $W = 2mn^2 - 2n^3/3$ to leading order. 7

5 (a) Show that for every $m \times n$ complex matrix A we can write: 15

$$A_1 \equiv U^* A V = \begin{bmatrix} \sigma & 0 \\ 0 & B \end{bmatrix} \tag{5}$$

where $\sigma = \|A\|$, B is an $(m-1) \times (n-1)$ matrix, $U = \begin{bmatrix} y_0 & U_1 \end{bmatrix}$, $V = \begin{bmatrix} x_0 & V_1 \end{bmatrix}$, the unit vector x_0 satisfies $\|Ax_0\| = \|A\|$ and finally $y_0 = \frac{Ax_0}{\|A\|}$. The matrices U_1 and V_1 are chosen so that U and V are unitary $m \times m$ and $n \times n$ respectively.

(The vector and induced matrix 2–norm is used throughout this question. You may assume that $\|OA\| = \|A\|$ for any unitary matrix O.)

(b) Use part (a) to prove by induction on the size of A that every $m \times n$ complex matrix A has a Singular Value Decomposition (SVD) $A = U\Sigma V^*$ where U is $m \times m$ unitary, V is $n \times n$ unitary and Σ is an $m \times n$ diagonal matrix of singular values. 10

6 For any $m \times m$ matrix A, Gauss Elimination without pivoting consists of:

Algorithm 0.2 *Gauss Elimination Without Pivoting — in words*

```
for k = 1 to m − 1
    Add suitable multiples of row k to the rows beneath
    to introduce zeroes below the main diagonal in column k.
end
```

(a) Show that each iteration of the above algorithm can be effected by left-multiplying A by a matrix $L_k = I - \ell_k e_k^*$ where ℓ_k is the vector of **multipliers** for the k^{th} column of A (the first k entries of ℓ_k are 0) and e_k is a vector in $\mathbb{C}^n$ with one in the k^{th} position and zeroes elsewhere. Give a simple formula for the non-zero entries of ℓ_k. 6

(b) Show that for each k, $L_k^{-1} = I + \ell_k e_k^*$. 1

(c) Show that the matrix $L = L_1{}^{-1} L_2{}^{-1} \dots L_{m-1}{}^{-1}$ is just $I + \ell_1 e_1^* + \dots + \ell_m e_m^*$. 2

(d) Explain briefly why the result in (c) means that L is lower triangular. 1

(e) What special structure does A have after the algorithm has completed? 1
(See the next page for the rest of Q.6.)

(f) When partial pivoting is applied, we have

$$L_{m-1}P_{m-1}L_{m-2}P_{m-2}\dots L_2P_2L_1P_1A = U$$

where each P_j swaps row j with one of the rows $j+1,\dots,m$ (if necessary) to make the absolute value of the "pivot" A_{jj} as large as possible. Define

$$\begin{aligned}\Pi_j &= P_{m-1}P_{m-2}\dots P_j, \quad j = 1,\dots,m-1\\ \Pi_m &= I\\ L'_j &= \Pi_{j+1}L_j\Pi_{j+1}^{-1}, \quad j = 1,\dots,m-1\end{aligned}$$

(i) First show that $\Pi_{j+1}^{-1}\Pi_j = P_j$. 2

(ii) Now show that

$$L'_{j+1}L'_j = \Pi_{j+2}L_{j+1}P_{j+1}L_j\Pi_{j+1}^{-1}$$

for any $j = 1,\dots,m-2$. 4

(iii) Finally, given (i) & (ii), show that: 6

$$L_{m-1}P_{m-1}L_{m-2}P_{m-2}\dots L_2P_2L_1P_1 = L'_{m-1}L'_{m-2}\dots L'_2L'_1\Pi_1.$$

(g) Show that the LU factorisation $A = LU$ (without pivoting) is now replaced by $PA = LU$ where $P = \Pi_1$ and $L = L_1'^{-1}L_2'^{-1}\dots L_{m-1}'^{-1}$. 2