UNIVERSITY of LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science & Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4105

SEMESTER: Autumn 2013

MODULE TITLE: Linear Algebra 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

PERCENTAGE OF TOTAL MARKS: 70%

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks, 70%.**

1 (a) Prove the following Lemma: Let V be a finite-dimensional vector space. Let $L = \{\underline{l}_1, \dots, \underline{l}_n\}$ be a linearly independent set in V and let $S = \{\underline{s}_1, \dots, \underline{s}_m\}$ be a second subset of V which spans V. Then $n \le m$. 15

Hint: Write each of the $\underline{l}_i$ in L as a linear combination of the spanning set S and use the fact that a homogeneous linear system $A^T c = 0$ with more unknowns than equations has non-trivial solutions where A is the coefficient matrix for expressing the vectors in L in terms of the vectors in S.

(b) Show that this result leads to a definition for the dimension of a vector space. 3

(c) Prove the Cauchy-Schwarz Inequality: if V is an real Inner Product Space, then for any $\underline{u}, \underline{v} \in V$, $\left|\langle \underline{u}, \underline{v} \rangle\right| \le \|\underline{u}\| \|\underline{v}\|$. 4

(d) Define $\cos\theta$, where θ is the angle between two vectors $\underline{u}, \underline{v}$ in an Inner Product Space V. Use the Cauchy-Schwarz Inequality to show that $|\cos\theta| \le 1$. 1

(e) Let V be an Inner Product Space with inner product $\langle ., . \rangle$ and $\|x\|^2 = \langle x, x \rangle$ for all $x \in V$. Prove the Parallelogram law :

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right)$$

holds in any inner product space V. Draw a sketch for the case where $V = \mathbb{R}^2$. 2

2 Given a choice of norm $\|\cdot\|$ on $\mathbb{C}^m$ and $\mathbb{C}^n$, the induced matrix norm of the $m \times n$ matrix A is defined as

$$\|A\| \equiv \sup_{\|x\|=1} \|Ax\|.$$

(a) Explain carefully why the two conditions 8

$$\|Ax\| \le B, \text{ for all unit vectors } x,$$
$$Ax_0 = B, \text{ for some unit vector } x_0$$

imply that $\|A\| = B$.

(b) Using the two conditions in part (a), show that the induced ∞–norm $\|A\|_\infty$ of a **real** $m \times n$ matrix A can be calculated as the "maximum row sum" of A:

$$\|A\|_\infty = \max_{1 \le i \le n} \|a_i^*\|_1.$$

where a_i^* stands for the i^{th} row of A. 10

(c) Use the result found in part (b) to calculate the induced ∞–norm of the 2×3 matrix $A = \begin{bmatrix} 4-i & 3+2i & 4 \\ 7-i & i & -3i \end{bmatrix}$. 2

(d) Again using the two conditions in part (a), show for any rank 1 $m \times n$ matrix $A = uv^*$ (where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$) that the induced 2–norm of A is given by

$$\|A\|_2 = \|u\|_2 \|v\|_2.$$

(Hint; check that the unit vector $x_0 = v/\|v\|$ attains the bound.) 5

3 Recall that a complex $m \times m$ matrix P is a projection operator if $P^2 = P$.

(a) Show that if λ is an eigenvalue of a projection operator P then either $\lambda = 0$ or $\lambda = 1$. 2

(b) Given an $m \times n$ complex matrix A with $m \geq n$, show that A^*A is invertible if and only if A has full rank (remember that a square matrix is invertible iff $Ax = 0$ implies $x = 0$.) 6

(c) Given a linearly independent set of vectors $\{a_1, \dots, a_n\}$ in $\mathbb{C}^m$, let A be the $m \times n$ matrix whose j^{th} column is a_j. Show using the result in (b) that P, the orthogonal projection operator onto the range of A, is given by the formula 6

$$P = A(A^*A)^{-1}A^*.$$

(d) Let A be an $m \times n$ complex matrix with $m \geq n$ and let $b \in \mathbb{C}^m$ be given. For any $x \in \mathbb{R}^n$, define the residual r by $r \equiv Ax - b$. Show that the following four conditions are equivalent (show that the first implies the second, etc.) 6

$$r \perp \text{range}(A) \tag{1}$$

$$A^*r = 0 \tag{2}$$

$$A^*Ax = A^*b \tag{3}$$

$$Pb = Ax \tag{4}$$

where $P \in \mathbb{C}^{m \times m}$ is the orthogonal projection operator onto the range of A.

(e) Show that $y = Pb$ is the vector in the range of A that minimises $\|b - z\|_2$ over all vectors $z \in \text{range}(A)$. 5

4 (a) For any vector $\nu \in \mathbb{C}^k$, let the matrix $H = I - 2P_\nu$ (where $P_\nu = \dfrac{\nu\nu^*}{\nu^*\nu}$);

(i) Show with a sketch that the effect on an arbitrary vector $x \in \mathbb{C}^k$ of left-multiplying x by H is to reflect x in $P_{\perp\nu}x$, the normal to ν in the x–ν plane. 3

(ii) Find the choices of vector ν that make Hx, the **Householder reflection** of x, return a multiple of e_1 where $e_1 \in \mathbb{C}^k$ is a vector of zeroes with one in the first position. 8

(iii) Which of the two choices found should be used and why? 1

(b) The following algorithm (Alg. 0.1) takes as its input an arbitrary $m \times n$ complex matrix A. Explain the effect of line 5 and relate it to the Householder reflection in part (a). 3

Algorithm 0.1 *Householder QR Factorisation*

(1) for $k = 1$ to n
(2) $\quad x = A_{k:m,k}$
(3) $\quad \nu_k = x + \text{sign}(x_1)\|x\|e_1$
(4) $\quad \nu_k = \nu_k / \|\nu_k\|$
(5) $\quad A_{k:m,k:n} = A_{k:m,k:n} - 2\nu_k\,(\nu_k^* A_{k:m,k:n})$
(6) end

(c) The work done in Alg. 0.1 is dominated by the (implicit) inner loop `j=k:n` over the columns of the submatrix $A_{k:m,k:n}$ in line 5. Show that the total operation count W for the algorithm is $W = 2mn^2 - 2/3n^3$ to leading order. 10

(Just show that the coefficient of n^3 in W is $-2/3$ and that the coefficient of mn^2 is 2.)

5 (a) Show that for every $m \times n$ complex matrix A we can write: 15

$$A_1 \equiv U^*AV = \begin{bmatrix} \sigma & 0 \\ 0 & B \end{bmatrix} \tag{5}$$

where $\sigma = \|A\|$, B is an $(m-1) \times (n-1)$ matrix, $U = \begin{bmatrix} y_0 & U_1 \end{bmatrix}$ and $V = \begin{bmatrix} x_0 & V_1 \end{bmatrix}$, the unit vector x_0 satisfies $\|Ax_0\| = \|A\|$ and finally $y_0 = \frac{Ax_0}{\|A\|}$. The matrices U_1 and V_1 are chosen so that U and V are unitary $m \times m$ and $n \times n$ respectively.

(The vector and induced matrix 2–norm is used throughout this question. You may assume that $\|OA\| = \|A\|$ for any unitary matrix O.)

(b) Using part (a) prove by induction on the size of A that every $m \times n$ complex matrix A has a Singular Value Decomposition (SVD) $A = U\Sigma V^*$ where U is $m \times m$ unitary, V is $n \times n$ unitary and Σ is an $m \times n$ diagonal matrix of singular values. 10

6 Thoughout this Question assume any $m \times n$ ($m \geq n$) complex matrix A has a Singular Value Decomposition (SVD)

$$A = U\Sigma V^* \tag{6}$$

where U and V are unitary $m \times m$ and $n \times n$ matrices respectively and Σ is an $m \times n$ diagonal matrix. The reduced SVD expresses A as $A = \hat{U}\hat{\Sigma}V^*$ where $\hat{U}$ consists of the first n columns of U and $\hat{\Sigma}$ the first n rows of Σ.

(a) Explain why the reduced SVD is equivalent to the full SVD. 2

(b) Show that for any $m \times n$ matrix A, the singular values σ_i (the diagonal elements of $\hat{\Sigma}$) may be found by computing the eigenvalues λ_i of A^*A and taking square roots — i.e. $\sigma_i = \sqrt{\lambda_i}$ and that the unitary $n \times n$ matrix V has the eigenvectors of A^*A as its columns. 4

(c) Derive the equation $AV = \hat{U}\hat{\Sigma}$ and explain how it may be used to find the reduced matrix $\hat{U}$. 3

(d) Given the matrix

$$A = \begin{pmatrix} 0 & -9 \\ -3 & 0 \\ 0 & 2 \end{pmatrix}$$

find a reduced SVD of A using any method you wish . (Remember that the singular values are in decreasing order: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.) 8

(e) Now find a **full** SVD for A using any method you wish. 3

(f) Show that any product of compatible matrices A and B can be written

$$AB = \sum_k a_k b_k^*$$

where a_k is the k^{th} column of A and b_k^* is the k^{th} row of B. 2

(g) Using (e) or otherwise show that any $m \times n$ matrix A can be written as a sum of rank 1 matrices:

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^*$$

where r is the rank of A (the number of non-zero singular values). 3