UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science & Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4105

SEMESTER: Autumn 2012

MODULE TITLE: Linear Algebra 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

PERCENTAGE OF TOTAL MARKS: 80%

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks, 80%.**

1 (a) Prove the following Lemma: Let V be a finite-dimensional vector space. Let $L = \{\underline{l}_1, \ldots, \underline{l}_n\}$ be a linearly independent set in V and let $S = \{\underline{s}_1, \ldots, \underline{s}_m\}$ be a second subset of V which spans V. Then $n \leq m$. 15%

Hint: Write each of the $\underline{l}_i$ in L as a linear combination of the spanning set S and use the fact that a homogeneous linear system $Ac = 0$ with more unknowns than equations has non-trivial solutions where A is the coefficient matrix for expressing the vectors in L in terms of the vectors in S.

(b) Show that this result leads to a definition for the dimension of a vector space. 3%

(c) The Cauchy-Schwarz Inequality states that if V is an Inner Product Space, then for any $\underline{u}, \underline{v} \in V$, $\left|\langle \underline{u}, \underline{v} \rangle\right| \leq \|\underline{u}\|\|\underline{v}\|$. Use the Cauchy-Schwarz Inequality to prove the Triangle Inequality: 3%

$$\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\| \quad \text{for all } \underline{u}, \underline{v} \in V.$$

(d) What relationship between $\underline{u}$ and $\underline{v}$ must hold for the left hand side in (c) equal to the right hand side? Justify your answer. 2%

(e) Show that

$$\|\underline{u} - \underline{v}\| \geq \|\underline{u}\| - \|\underline{v}\| \quad \text{for all } \underline{u}, \underline{v} \in V.$$

follows from (c). 2%

2 (a) Suppose that an $n \times n$ real matrix A has full rank (the columns of A are linearly independent). Show that A has an inverse. 12%

(b) Show for any compatible (matrices for which the product AB is defined) matrices A and B that $(AB)^* = B^*A^*$. 2%

(c) Use (b) to show that for any invertible matrix A, $(A^*)^{-1} = (A^{-1})^*$. 2%

(d) Suppose that the complex matrix S is skew-hermitian ($S^* = -S$ where S^* is the hermitian conjugate of S). Show that the matrix S must have pure imaginary eigenvalues. 4%

(e) Show that the matrix $I - S$ must be invertible given that S is skew-hermitian. (Use the result from 2(d) .) 4%

(f) A unitary matrix Q satisfies $Q^*Q = I$. Must Q also satisfy $QQ^* = I$? Explain briefly. 1%

3 Thoughout this Question we have that any $m \times n$ $(m \geq n)$ complex matrix A has a Singular Value Decomposition (SVD)

$$A = U\Sigma V^* \tag{1}$$

where U and V are unitary $m \times m$ and $n \times n$ matrices respectively and Σ is an $m \times n$ diagonal matrix. The reduced SVD expresses A as $A = \hat{U}\hat{\Sigma}V^*$ where $\hat{U}$ consists of the first n columns of U and $\hat{\Sigma}$ the first n rows of Σ.

(a) Explain why the reduced SVD is equivalent to the full SVD. 2%

(b) Show that for any $m \times n$ matrix A, the singular values σ_i (the diagonal elements of $\hat{\Sigma}$) may be found by computing the eigenvalues λ_i of A^*A and taking square roots — i.e. $\sigma_i = \sqrt{\lambda_i}$ and that the unitary $n \times n$ matrix V has the eigenvectors of A^*A as its columns. 4%

(c) Derive the equation $AV = \hat{U}\hat{\Sigma}$ and explain how it may be used to find the reduced matrix $\hat{U}$. 2%

(d) Given the matrix

$$A = \begin{pmatrix} 0 & 7 \\ 4 & 0 \\ 0 & -5 \end{pmatrix}$$

find a reduced SVD of A using any method you wish . 10%

(e) Now find a **full** SVD for A using any method you wish. 2%

(f) Show that any product of compatible matrices A and B can be written

$$AB = \sum_k a_k b_k^*$$

where a_k is the k^{th} column of A and b_k^* is the k^{th} row of B. 2%

(g) Using (e) or otherwise show that any $m \times n$ matrix A can be written as a sum of rank-1 matrices:

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^*$$

where r is the rank of A (the number of non-zero singular values). 3%

4 Recall that a complex $m \times m$ matrix P is a projection operator if $P^2 = P$.

(a) Show that if λ is an eigenvalue of a projection operator P then either $\lambda = 0$ or $\lambda = 1$. 2%

(b) We say that S_1 and S_2, a pair of subspaces of $\mathbb{C}^m$, are **complementary** if $S_1 \cap S_2 = \{\mathbf{0}\}$ and every $x \in \mathbb{C}^m$ can be written as a linear combination of vectors from S_1 and S_2. Show that if S_1 and S_2 are complementary we can always find a projection operator P such that range $P = S_1$ and null $P = S_2$. 4%

(c) If a pair of complementary subspaces S_1 and S_2 are orthogonal (every vector from S_1 is orthogonal to every vector from S_2) then we say that the corresponding projection operator P is orthogonal. Show that a projection operator P is orthogonal iff $P = P^*$. 8%

(d) Given an $m \times n$ complex matrix A with $m \geq n$, show that A^*A is invertible if and only if A has full rank (remember that rank $A =$ rank A^*). 5%

(e) Given a linearly independent set of vectors $\{a_1, \ldots, a_n\}$ in $\mathbb{C}^m$, let A be the $m \times n$ matrix whose j^{th} column is a_j. Show that P, the orthogonal projection operator onto the range of A, is given by the formula 6%

$$P = A(A^*A)^{-1}A^*.$$

5 (a) For any vector $v \in \mathbb{C}^k$, let the matrix $H = I - 2P_v$ (where $P_v = \dfrac{vv^*}{v^*v}$);

(i) Show with a sketch that the effect on an arbitrary vector $x \in \mathbb{C}^k$ of left-multiplying x by H is to reflect x in $P_{\perp v}x$, the normal to v in the x–v plane. 3%

(ii) Find the choices of vector v that make Hx, the **Householder reflection** of x, return a multiple of e_1 where $e_1 \in \mathbb{C}^k$ is a vector of zeroes with one in the first position. 10%

(iii) Which of the two choices found should be used and why? 1%

(See over for the rest of Q.5.)

(b) The following algorithm (Alg. 0.1) takes as its input an arbitrary $m \times n$ complex matrix A. Explain the effect of line 5 and relate it to the Householder reflection in part (a). 3%

Algorithm 0.1 *Householder QR Factorisation*

```
(1) for k = 1 to n
(2)     x = A_{k:m,k}
(3)     v_k = x + sign(x_1)||x||e_1
(4)     v_k = v_k/||v_k||
(5)     A_{k:m,k:n} = A_{k:m,k:n} − 2v_k (v_k^* A_{k:m,k:n})
(6) end
```

(c) The work done in Alg. 0.1 is dominated by the (implicit) inner loop `j=k:n` over the columns of the submatrix $A_{k:m,k:n}$ in line 5. Show that the operation count for the algorithm is $2mn^2 - 2/3n^3$ to leading order. 8%

(Just show that the coefficient of n^3 in your total is $-2/3$ and that the coefficient of mn^2 is 2.)

6 For any $m \times m$ matrix A, Gauss Elimination without pivoting consists of:

Algorithm 0.2 *Gauss Elimination Without Pivoting — in words*

```
for k = 1 to m − 1
    Add suitable multiples of row k to the rows beneath
    to introduce zeroes below the main diagonal in column k.
end
```

(a) Show that each iteration of the above algorithm can be effected by left-multiplying A by a matrix $L_k = I - \ell_k e_k^*$ where ℓ_k is the vector of **multipliers** for the k^{th} column of A (the first k entries of ℓ_k are 0) and e_k is a vector in $\mathbb{C}^n$ with one in the k^{th} position and zeroes elsewhere. Give a simple formula for the non-zero entries of ℓ_k. 6%

(b) Show that for each k, $L_k^{-1} = I + \ell_k e_k^*$. 2%

(c) Show that the matrix $L = L_1{}^{-1} L_2{}^{-1} \dots L_{m-1}{}^{-1}$ is just $I + \ell_1 e_1^* + \cdots + \ell_m e_m^*$. 2%

(d) Explain briefly why the result in (c) means that L is lower triangular. 2%

(e) What special structure does A have after the algorithm has completed? 1%
(See over for the rest of Q.6.)

(f) When partial pivoting is applied, we have

$$L_{m-1}P_{m-1}L_{m-2}P_{m-2}\dots L_2P_2L_1P_1A = U$$

where each P_j swaps row j with one of the rows $j+1,\dots,m$ (if necessary) to make the absolute value of the "pivot" A_{jj} as large as possible. Defining

$$\begin{aligned} \Pi_j &= P_{m-1}P_{m-2}\dots P_j \\ L'_j &= \Pi_{j+1}L_j\Pi_{j+1}^{-1}, \end{aligned}$$

show that 6%

$$L_{m-1}P_{m-1}L_{m-2}P_{m-2}\dots L_2P_2L_1P_1 = L'_{m-1}L'_{m-2}\dots L'_2L'_1 \quad \Pi_1.$$

(g) Show that the LU factorisation $A = LU$ (without pivoting) is now replaced by $PA = LU$ (with pivoting) where $P = \Pi_1$ and $L = L_1'^{-1}L_2'^{-1}\dots L_{m-1}'^{-1}$. 1%

(h) Finally, show that the matrix L is lower triangular as it was in the no-pivoting case. 5%