UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science & Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4105

SEMESTER: Autumn 2010

MODULE TITLE: Linear Algebra 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

PERCENTAGE OF TOTAL MARKS: 70%

EXTERNAL EXAMINER: Dr. T. Myers

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks, 70%.**

1 (a) Let V be a finite-dimensional vector space. Let $L = \{l_1, \ldots, l_n\}$ be a linearly independent set in V and let $S = \{s_1, \ldots, s_m\}$ be a second subset of V which spans V. Prove that $n \leq m$. 15%

(Hint: a homogeneous linear system with more unknowns than equations has non-trivial (not all components zero) solutions.)

(b) Explain briefly how this result leads to a definition for the dimension of a vector space. 3%

(c) If u and v are vectors in a real inner product space then prove the Cauchy-Schwarz inequality: 6%

$$|< u, v >| \leq \|u\|\|v\|.$$

(d) What is the significance of the Cauchy-Schwarz inequality for the cosine of the angle between two vectors in a real inner product space? 1%

2 (a) Suppose that an $n \times n$ real matrix A has full rank (the columns of A are linearly independent). Show that A is invertible. (You may assume that the columns of A span $\mathbb{R}^n$.) 12%

(b) Show for any compatible (matrices for which the product AB is defined) matrices A and B that $(AB)^* = B^*A^*$. 3%

(c) Show that for any invertible matrix A, $(A^*)^{-1} = (A^{-1})^*$. 3%

(d) Suppose that the complex matrix S is skew-hermitian ($S^* = -S$ where S^* is the hermitian conjugate of S). Show that the matrix

$$Q = (I - S)^{-1}(I + S)$$

is unitary ($Q^*Q = I$). (Use the results 2(b) and 2(c).) 6%

(e) Does a unitary matrix always have the property $QQ^* = I$? Explain briefly. 1%

3 (a) Show that for every $m \times n$ complex matrix A we can write: 15%

$$A_1 \equiv U^*AV = \begin{bmatrix} \sigma & 0 \\ 0 & B \end{bmatrix} \qquad (1)$$

where $\sigma = \|A\|_2$, B is an $(m-1) \times (n-1)$ matrix, $U = \begin{bmatrix} y_0 & U_1 \end{bmatrix}$ and $V = \begin{bmatrix} x_0 & V_1 \end{bmatrix}$, the unit vector x_0 satisfies $\|Ax_0\| = \|A\|$ and finally $y_0 = \frac{Ax_0}{\|A\|}$. The matrices U_1 and V_1 are chosen so that U and V are unitary $m \times m$ and $n \times n$ respectively. (You may assume that $\|OA\| = \|A\|$ for any unitary matrix O.)

(b) Using part (a) prove by induction on the size of A that every $m \times n$ complex matrix A has a Singular Value Decomposition (SVD) $A = U\Sigma V^*$ where U is $m \times m$ unitary, V is $n \times n$ unitary and Σ is an $m \times n$ diagonal matrix of singular values. 10%

4 Recall that a complex $n \times n$ matrix P is a projection operator if $P^2 = P$.

(a) Given a projection operator P, show that any vector in $x \in \mathbb{R}^n$ can be written as $x = x_1 + x_2$ where $x_1 \in \text{range}\, P$ and $x_2 \in \text{null}\, P$. 2%

(b) Show that for any projection operator P, $\text{range}(I - P) = \text{null}(P)$. 4%

(c) Show that if P is a projection operator then so is $I - P$ and conclude from the result of part (b) that $\text{range}\, P = \text{null}(I - P)$. 1%

(d) We say that S_1 and S_2, a pair of subspaces of $\mathbb{C}^m$, are **complementary** if $S_1 \cap S_2 = \{0\}$ and every $x \in \mathbb{C}^m$ can be written as a linear combination of vectors from S_1 and S_2. Show that given S_1 and S_2 complementary we can always find a projection operator P such that $\text{range}\, P = S_1$ and $\text{null}\, P = S_2$. 4%

(e) If a pair of complementary subspaces S_1 and S_2 are **orthogonal** (every vector from S_1 is orthogonal to every vector from S_2) then we say that the corresponding projection operator P is orthogonal. Show that a projection operator P is orthogonal iff $P = P^*$. 4%+10%

5 (a) For any vector $v \in \mathbb{C}^k$, let the matrix $H = I - 2P_v$ (where $P_v = \dfrac{vv^*}{v^*v}$);

(i) Show with a sketch that the effect on an arbitrary vector $x \in \mathbb{C}^k$ of left-multiplying x by H is to reflect x in $P_{\perp v}x$, the normal to v in the x–v plane. 3%

(ii) Find the choices of vector v that make Hx, the **Householder reflection** of x, return a multiple of e_1 where $e_1 \in \mathbb{C}^k$ is a vector of zeroes with one in the first position. 10%

(iii) Which of the two choices found should be used and why? 1%

(b) The following algorithm (Alg. 0.1) takes as its input an arbitrary $m \times n$ complex matrix A. Explain the effect of line 5 and relate it to the Householder reflection in part (a). 3%

Algorithm 0.1 *Householder QR Factorisation*

```
(1) for k = 1 to n
(2)     x = A_{k:m,k}
(3)     v_k = x + sign(x_1)||x||e_1
(4)     v_k = v_k/||v_k||
(5)     A_{k:m,k:n} = A_{k:m,k:n} − 2v_k (v_k^* A_{k:m,k:n})
(6) end
```

(c) The work done in Alg. 0.1 is dominated by the (implicit) inner loop `j=k:n` over the columns of the submatrix $A_{k:m,k:n}$ in line 5. Show that the operation count for the algorithm is $2mn^2 - 2/3n^3$ to leading order. 8%

(Just show that the coefficient of n^3 in your total is $-2/3$ and that the coefficient of mn^2 is 2.)

6 For any $m \times m$ matrix A, Gauss Elimination without pivoting consists of:

Algorithm 0.2 *Gauss Elimination Without Pivoting — in words*

```
for k = 1 to m − 1
    Add suitable multiples of row k to the rows beneath
    to introduce zeroes below the main diagonal in column k.
end
```

(a) Show that each iteration of the above algorithm can be effected by left-multiplying A by a matrix $L_k = I - \ell_k e_k^*$ where ℓ_k is the vector of **multipliers** for the k^{th} column (the first k entries of ℓ_k are 0) and e_k is a vector in $\mathbb{C}^n$ with one in the k^{th} position and zeroes elsewhere. Give a simple formula for the non-zero entries of ℓ_k. 6%

(b) Show that for each k, $L_k^{-1} = I + \ell_k e_k^*$. 3%

(c) Show that the matrix $L = {L_1}^{-1}{L_2}^{-1} \dots {L_{m-1}}^{-1}$ is just $I + \ell_1 e_1^* + \cdots + \ell_m e_m^*$. 3%

(d) What special structure does L have? 2%

(e) What special structure does A have after the algorithm has completed? 1%

(See over for the rest of Q.6.)

(f) Explain briefly the operation of the following implementation of Gauss Elimination, algorithm (Alg. 0.3) which takes as its input an arbitrary matrix $m \times m$ complex matrix A, in particular, how does the inner loop (lines 3–6) correspond to multiplication by L_k? 2%

Algorithm 0.3 *Gauss Elimination without Pivoting*

```
(1)  U = A, L = I
(2)  for k = 1 to m − 1
(3)      for j = k + 1 to m
(4)          ℓ_jk = u_jk / u_kk
(5)          u_{j,k:m} = u_{j,k:m} − ℓ_jk u_{k,k:m}
(6)      end
(7)  end
```

(g) The work done in Alg. 0.3 is dominated by Line 5. Show that the operation count for the algorithm is $\approx 2/3m^3$.
(Just find the coefficient of m^3 in your total.) 8%