UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science & Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4105

SEMESTER: Autumn 2009

MODULE TITLE: Linear Algebra 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

PERCENTAGE OF TOTAL MARKS: 80%

EXTERNAL EXAMINER: Dr. P. Howell

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks, 80%.**

1 (a) Let V be a finite-dimensional vector space. Let $L = \{l_1, \ldots, l_n\}$ be a linearly independent set in V and let $S = \{s_1, \ldots, s_m\}$ be a second subset of V which spans V. Prove that $n \leq m$. 13%

(Hint: a homogeneous linear system with more unknowns than equations has non-trivial (not all components zero) solutions.)

(b) Explain briefly how this result leads to a definition for the dimension of a vector space. 2%

(c) Let V be any non-zero finite-dimensional inner product space and suppose that $\{v_1, \ldots, v_n\}$ is a basis for V. To compute an orthonormal basis $\{u_1, \ldots, u_n\}$ we can use the Gram-Schmidt (G-S) Orthogonalisation Process algorithm :

Algorithm 0.1

Gram-Schmidt Orthogonalisation Process

$u_1 = v_1/\|v_1\|$

while $(i \leq n)$ **do**

$$u_i = v_i - \sum_{j=1}^{i-1} \langle v_i, u_j \rangle u_j$$

$u_i = u_i/\|u_i\|$

end

Use the G–S Process to compute an orthonormal basis for the inner product space spanned by the polynomials $\{1, x, x^2\}$ using the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$. 10%

2 (a) Show that if S is a set of n linearly independent vectors in an n-dimensional vector space then S must also span V — so that S is a basis for V. 7%

(b) Suppose that an $n \times n$ real matrix A has full rank (the columns of A are linearly independent). Show that A is invertible. (Use the result from 2.(a)). 7%

(c) Show for any compatible complex matrices A and B that $(AB)^* = B^*A^*$. 3%

(d) Show for any compatible complex square matrices A and B that $(AB)^{-1} = B^{-1}A^{-1}$. 1%

(e) Defining the induced norm of an $m \times n$ complex matrix $A$by

$$\|A\| = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|Ax\|}{\|x\|}$$

show that the 1-norm of A is the "maximum column sum" of A, i.e. that 7%

$$\|A\|_1 = \max_{1 \leq j \leq n} \|a_j\|_1,$$

where a_j are the columns of A and the 1-norm of a vector x is $\|x\| = |x_1| + \cdots + |x_m|$.

3 Thoughout this Question we have that any $m \times n$ ($m \geq n$) complex matrix A has a Singular Value Decomposition (SVD)

$$A = U\Sigma V^* \tag{1}$$

where U and V are unitary $m \times m$ and $n \times n$ matrices respectively and Σ is an $m \times n$ diagonal matrix. The reduced SVD expresses A as $A = \hat{U}\hat{\Sigma}V^*$ where $\hat{U}$ consists of the first n columns of U and $\hat{\Sigma}$ the first n rows of Σ.

(a) Explain why the reduced SVD is equivalent to the full SVD. 2%

(b) Consider the $n \times n$ matrix A^*A. 1%,2%

(i) Show that for any $m \times n$ matrix A, the singular values (diagonal elements of $\hat{\Sigma}$) may be found by computing the eigenvalues of A^*A.

(ii) Show that the unitary $n \times n$ matrix V has the eigenvectors of A^*A as its columns.

(c) How can the matrix equation $AV = \hat{U}\hat{\Sigma}$ be used to find the reduced matrix $\hat{U}$? 4%

(d) Given the matrix

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$$

find a reduced SVD of A using any method you wish . 10%

(e) Show that any $m \times n$ matrix A can be written as a sum of rank-1 matrices:

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^*$$

where r is the rank of A (the number of non-zero singular values). 4%

(f) Explain briefly how this result may be used to compress a digital image. 2%

4 Recall that a complex $m \times m$ matrix P is a projection operator if $P^2 = P$.

(a) Show that for any projection operator P, $\text{range}(I - P) = \text{null}(P)$. 3%

(b) We say that S_1 and S_2, a pair of subspaces of $\mathbb{C}^m$, are **complementary** if $S_1 \cap S_2 = \{0\}$ and every non-zero $x \in \mathbb{C}^m$ is in exactly one of S_1 or S_2. Show that given S_1 and S_2 complementary we can always find a projection operator P such that range $P = S_1$ and null $P = S_2$. 2%

(c) If a pair of complementary subspaces S_1 and S_2 are orthogonal (every vector from S_1 is orthogonal to every vector from S_2) then we say that the corresponding projection operator P is orthogonal. Show that a projection operator P is orthogonal iff $P = P^*$. 6%

(d) Given an $m \times n$ complex matrix A with $m \geq n$, show that A^*A is invertible if and only if A has full rank. 7%

(e) Given a linearly independent set $\{a_1, \ldots, a_n\}$ let A be the $m \times n$ matrix whose j^{th} column is a_j. Show that P, the orthogonal projection operator onto the range of A, is given by the formula 7%

$$P = A(A^*A)^{-1}A^*.$$

5 (a) For any vector $v \in \mathbb{C}^p$, let the matrix $H = I - 2P_v$ (where $P_v = \dfrac{vv^*}{v^*v}$);

(i) Show with a sketch that the effect on an arbitrary vector $x \in \mathbb{C}^p$ of left-multiplying x by H is to reflect x in the normal to v in the x–v plane. 3%

(ii) Find the choices of vector v that make Hx, the **Householder reflection** of x, return a multiple of e_1 where $e_1 \in \mathbb{C}^p$ is a vector of zeroes with one in the first position. 9%

(iii) Which of the two choices found should be used and why? 1%

(b) Explain the operation of the following algorithm (Alg. 0.2) which takes as its input an arbitrary matrix $m \times n$ complex matrix A.

(i) Explain how the choice of v_k in lines 3 & 4 relates to the vector v found in part (a). 1%

(ii) Explain the effect of line 5 and relate it to the Householder reflection in part (a). 3%

(iii) What special structure will the matrix A have after the algorithm has completed? 1%

Algorithm 0.2 *Householder QR Factorisation*

1 **for** $k = 1$ **to** n
2 $\quad x = A_{k:m,k}$
3 $\quad v_k = x + \text{sign}(x_1)\|x\|e_1$
4 $\quad v_k = v_k / \|v_k\|$
5 $\quad A_{k:m,k:n} = A_{k:m,k:n} - 2v_k\left(v_k^* A_{k:m,k:n}\right)$
6 **end**

(c) The work done in Alg. 0.2 is dominated by the (implicit) inner loop `j=k:n` over the columns of the submatrix $A_{k:m,k:n}$ in line 5. Show that the operation count for the algorithm is $2mn^2 - 2/3n^3$ to leading order. 7%

(Just show that the coefficient of n^3 in your total is $-2/3$ and that the coefficient of mn^2 is 2.)

6 For any $m \times m$ matrix A, Gauss Elimination without pivoting consists of:

Algorithm 0.3 *Gauss Elimination Without Pivoting — in words*

for $k = 1$ **to** $m - 1$
$\quad$ Add suitable multiples of row k to the rows beneath
$\quad$ to introduce zeroes below the main diagonal in column k.
end

(a) Show that each iteration of the above algorithm can be effected by left-multiplying A by a matrix $L_k = I - \ell_k e_k^*$ where ℓ_k is the vector of **multipliers** for the k^{th} column (the first k entries of ℓ_k are 0) and e_k is a vector in $\mathbb{C}^n$ with one in the k^{th} position and zeroes elsewhere. Give a simple formula for the non-zero entries of ℓ_k. 6%

(b) Show that for each k, $L_k^{-1} = I + \ell_k e_k^*$. 3%

(c) Show that the matrix $L = L_1{}^{-1} L_2{}^{-1} \dots L_{m-1}{}^{-1}$ is just the identity matrix I with the non-zero parts of each of the multiplier vectors $\ell_1, \ell_2, \dots$ inserted under the main diagonal in columns $1, 2, \dots$. 5%

(d) What special structure does L have? 1%

(e) What special structure does A have after the algorithm has completed? 1%

(f) Explain the operation of the following implementation of Gauss Elimination, algorithm (Alg. 0.4) which takes as its input an arbitrary matrix $m \times m$ complex matrix A, in particular, how does the inner loop (lines 3–6) correspond to multiplication by L_k? 3%

Algorithm 0.4 *Gauss Elimination without Pivoting*

```
1  U = A, L = I
2  for k = 1 to m − 1
3      for j = k + 1 to m
4          ℓ_jk = u_jk / u_kk
5          u_{j,k:m} = u_{j,k:m} − ℓ_jk u_{k,k:m}
6      end
7  end
```

(g) The work done in Alg. 0.4 is dominated by Line 5. Show that the operation count for the algorithm is $\approx 2/3m^3$.
(Just find the coefficient of m^3 in your total.) 6%