UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science & Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4105

SEMESTER: Autumn 2008/9

MODULE TITLE: Linear Algebra 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

PERCENTAGE OF TOTAL MARKS: 100%

EXTERNAL EXAMINER: Prof. J. Flavin

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks, 100%.**

1 (a) Let V be a finite-dimensional vector space and let $\{v_1, \dots, v_n\}$ be a basis for V. Prove **one** of the following propositions (**A or B**): 13%

(A) If a set of vectors in V has more than n vectors, it is linearly dependent.

(B) If a set of vectors in V has fewer than n vectors, it cannot span V.

You may assume that a homogeneous linear system with more unknowns than equations has infinitely many solutions.

(b) Explain briefly how results (A) and (B) lead to a definition for the dimension of a vector space. 2%

(c) If u and v are vectors in a real inner product space then prove the Cauchy-Schwarz inequality: 6%

$$|< u, v >| \leq \|u\|\|v\|. \tag{1}$$

(d) Use the Cauchy-Schwarz inequality to derive the Triangle Inequality: 4%

$$\|u + v\| \leq \|u\| + \|v\| \tag{2}$$

2 (a) Suppose that an $n \times n$ real matrix A has full rank (the columns of A are linearly independent). Show that A is invertible. (You may assume that the columns of A span $\mathbb{R}^n$.) 6%

(b) Suppose that the complex matrix S is skew-hermitian ($S^* = -S$ where S^* is the hermitian conjugate of S). Show that the matrix

$$Q = (I - S)^{-1}(I + S)$$

is unitary ($Q^*Q = I$). 5%

(c) Does a unitary matrix always have the property $QQ^* = I$? Explain briefly. 1%

(d) Show that for any vector $x \in \mathbb{C}^m$; 2%+2%

(i) $\|x\|_\infty \leq \|x\|_2$,

(ii) $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$,

(e) Using the general definition of an induced matrix norm

$$\|A\| = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|Ax\|}{\|x\|},$$

show using the results from 2.(d) that for any $m \times n$ complex matrix A; 5%+4%

(i) $\|A\|_\infty \leq \sqrt{n}\|A\|_2$,

(ii) $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$.

3 (a) Show that for every $m \times n$ complex matrix A we can write: 10%

$$A_1 \equiv U^*AV = \begin{bmatrix} \sigma & 0 \\ 0 & B \end{bmatrix} \qquad (3)$$

where $\sigma = \|A\|_2$, B is an $(m-1) \times (n-1)$ matrix, $U = \begin{bmatrix} y_0 & U_1 \end{bmatrix}$ and $V = \begin{bmatrix} x_0 & V_1 \end{bmatrix}$, the unit vector $x0$ satisfies $\|Ax_0\| = \|A\|$ and finally $y_0 = \frac{Ax_0}{\|A\|}$. The matrices U_1 and V_1 are chosen so that U and V are unitary $m \times m$ and $n \times n$ respectively.

(b) Using part (a) prove by induction on the size of A that every $m \times n$ complex matrix A has a Singular Value Decomposition (SVD) $A = U\Sigma V^*$ where U is $m \times m$ unitary, V is $n \times n$ unitary and Σ, is an $m \times n$ diagonal matrix of singular values. 8%

(c) Use the SVD to show that the row and column ranks of any $m \times n$ matrix are equal to r, the number of non-zero singular values. (Assume that $m > n$.) 7%

4 (a) For any vector $v \in \mathbb{C}^p$, let the matrix $H = I - 2P_v$ (where $P = \dfrac{vv^*}{v^*v}$);

(i) Show with a sketch that the effect on an arbitrary vector $x \in \mathbb{C}^p$ of left-multiplying x by H is to reflect x in the normal to v in the x–v plane. 3%

(ii) Find the choices of vector v that result in the **Householder reflection** of x, Hx, being a multiple of e_1 where $e_1 \in \mathbb{C}^p$ is a vector of zeroes with one in the first position. 7%

(iii) Which of the two choices found should be used and why? 1%

(iv) Given the vector $x = (-1, 3, 1, -2)^T \in \mathbb{R}^4$, find the appropriate choice of v that ensures that Hx is a multiple of e_1. 2%

(v) With this choice of v, what is the value of Hx?

(b) Explain the operation of the following algorithm (Alg. 0.1) which takes as its input an arbitrary matrix $m \times n$ complex matrix A.

(i) Explain how the choice of v_k in lines 3 & 4 relates to that found in part (a). 1%

(ii) Explain the effect of line 5 and relate it to the Householder reflection in part (a). 3%

(iii) What special structure will the matrix A have after the algorithm has completed? 1%

Algorithm 0.1 *Householder QR Factorisation*

1 **for** $k = 1$ **to** n
2 $\quad x = A_{k:m,k}$
3 $\quad v_k = x + \text{sign}(x_1)\|x\|e_1$
4 $\quad v_k = v_k/\|v_k\|$
5 $\quad A_{k:m,k:n} = A_{k:m,k:n} - 2v_k\,(v_k^* A_{k:m,k:n})$
6 **end**

(c) The work done in Alg. 0.1 is dominated by the (implicit) inner loop `j=k:n` over the columns of the submatrix $A_{k:m,k:n}$ in line 5. Show that the operation count for the algorithm is $\approx 2mn^2 - 2/3n^3$. 7%

5 For any $m \times m$ matrix A, Gauss Elimination without pivoting consists of:

Algorithm 0.2 *Gauss Elimination Without Pivoting — in words*

for $k = 1$ **to** $m - 1$
$\quad$ Add suitable multiples of row k to the rows beneath
$\quad$ to introduce zeroes below the main diagonal in column k.
end

(a) Show that each iteration of the above algorithm can be effected by left-multiplying A by a matrix $L_k = I - \ell_k e_k^*$ where ℓ_k is the vector of **multipliers** for the k^{th} column (the first k entries of ℓ_k are 0) and e_k is a vector in $\mathbb{C}^n$ with one in the k^{th} position and zeroes elsewhere. Give a simple formula for the non-zero entries of ℓ_k. 6%

(b) Show that each L_k can be inverted by negating its sub-diagonal elements. 3%

(c) Show that the product of the $m - 1$ matrices $L_k{}^{-1}$ in increasing order of k is the identity matrix with the non-zero sub-diagonal elements of the $L_k{}^{-1}$ inserted in the appropriate places. 3%

(d) What special structure does A have after the algorithm has completed? 1%

(e) Explain the operation of the following implementation of Gauss Elimination, algorithm (Alg. 0.3) which takes as its input an arbitrary matrix $m \times m$ complex matrix A, in particular, how does the inner loop (lines 3–6) correspond to multiplication by L_k? 3%

Algorithm 0.3 *Gauss Elimination without Pivoting*

```
1  U = A, L = I
2  for k = 1 to m − 1
3      for j = k + 1 to m
4          ℓ_jk = u_jk / u_kk
5          u_{j,k:m} = u_{j,k:m} − ℓ_jk u_{k,k:m}
6      end
7  end
```

(f) The work done in Alg. 0.3 is dominated by Line 5. Show that the operation count for the algorithm is $\approx 2/3m^3$. 6%

(g) Explain briefly how partial pivoting works, when it must be used and why it is always a good strategy. 3%

6 (a) Show that every $m \times m$ matrix A has the property that: 4%

$$Q_0^* A Q_0 = \begin{bmatrix} \lambda_1 & \nu_1^* \\ 0 & A_1 \end{bmatrix}, \qquad (4)$$

where $Q_0 = \begin{bmatrix} q_1 & q_2' & \dots & q_n' \end{bmatrix}$, q_1 is an eigenvector of A, λ_1 is the corresponding eigenvalue and the vectors $q_2', \dots, q_n'$ are chosen so that $\{q_1, q_2', \dots, q_n'\}$ form an orthonormal basis for $\mathbb{C}^n$. (Also $\nu_1 \in C^{n-1}$ and $A_1 \in \mathbb{C}^{(m-1)\times(m-1)}$.)

(b) Use induction to prove that every $m \times m$ complex matrix A has a Schur decomposition: 6%

$$A = QTQ^* \qquad (5)$$

where Q is unitary and T is upper triangular.

(c) Show that, given the Schur decomposition $A = QTQ^*$, the eigenvalues of A appear on the main diagonal of T. 8%

(d) Explain briefly why an **exact** method (one that calculates the eigenvalues in a finite number of steps in exact arithmetic) for calculating the eigenvalues of a matrix A is impossible. 1%

(e) The following algorithm (Alg. 0.4) reduces any $m \times m$ matrix A to Hessenberg form using Householder reflections by multiplying A successively on the left by Q_k^* and on the right by Q_k, $k = 1, \ldots, m-2$. The matrix $Q_k = \begin{bmatrix} I_k & 0 \\ 0 & H_k \end{bmatrix}$, I_k is the $k \times k$ identity matrix and H_k is the Householder reflector (see Q.4(a)) that transforms the vector $x_k = A_{k+1:m,k}$ into a multiple of e_1 at each iteration.

Explain carefully how lines 5 and 6 of Alg. 0.4 correspond to left-multiplication by Q_k^* and right-multiplication by Q_k respectively (note that the matrices Q_k are hermitian, i.e. $Q_k^* = Q_k$): 6%

Algorithm 0.4 *Householder Reduction to Hessenberg Form*

1 **for** $k = 1$ **to** $m - 2$

2 $\quad x = A_{k+1:m,k}$

3 $\quad v_k = x + \text{sign}(x_1)\|x\|e_1$

4 $\quad v_k = v_k / \|v_k\|$

5 $\quad A_{k+1:m,k:m} = A_{k+1:m,k:m} - 2v_k\left(v_k^* A_{k+1:m,k:m}\right)$

6 $\quad A_{1:m,k+1:m} = A_{1:m,k+1:m} - 2\left(A_{1:m,k+1:m} v_k\right) v_k^*$

7 **end**