UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4601

SEMESTER: Autumn 2003

MODULE TITLE: Science Mathematics 1

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. J. Kinsella

PERCENTAGE OF TOTAL MARKS: 80%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks — 80%.**

1 (a) Given the vectors $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 3, 2, 1\rangle$, find

(i) $|\mathbf{a}|$, 1

(ii) $|\mathbf{b}|$, 1

(iii) $\mathbf{a} \cdot \mathbf{b}$, 1

(iv) the angle between $\mathbf{a}$ and $\mathbf{b}$. (No calculator needed.). 1

(b) (i) Show that if θ is the angle between two vectors $\mathbf{a}$ & $\mathbf{b}$ in V^3 (so that $0 \le \theta \le \pi$), then 10

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta.$$

(ii) Find the area of the triangle PQR where $P = (1, 2, 4)$, $Q = (2, 3, 5)$ and $R = (0, 1, 0)$. 8

(iii) Find a vector perpendicular to the plane in which the triangle PQR lies. 3

2 (a) Find the determinant of the matrix 6

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

(b) What is the significance of the value you found for the determinant of A in 2(a)? 3

(c) Using either Gauss Elimination or Gauss-Jordan elimination, solve the linear system: 16

$$\begin{aligned} 2x + 3y &= -1 \\ x + 2z &= 3 \\ 2y + z &= -1 \end{aligned}$$

3 To find a straight line $y = mx + c$ passing through a set of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ we need to solve the set of linear equations:

$$\begin{aligned} y_1 &= mx_1 + c \\ y_2 &= mx_2 + c \\ &\vdots \\ y_n &= mx_n + c \end{aligned}$$

for the two unknowns m, c.

This linear system can be written as

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{1}$$

or just

$$M\mathbf{a} = \mathbf{y} \tag{2}$$

where $M = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} c \\ m \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

(a) Explain briefly why Equation (1) cannot be solved directly for c and m. 3

(b) Show that multiplying (2) on each side by the transpose of the matrix M gives the matrix equation: 8

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix}.$$

(c) A researcher monitors the voltage V in an electric circuit as the temperature T varies.

T	-1	0	1	2	3	4
V	-5	-3	-1	1	3	4

Find the least squares straight line fit to the data in the form $V = mT + c$. 12

(d) Use the least squares fit found in part 3(c) above to predict the voltage V in the circuit when $T = 6.5$. 2

4 (a) Calculate the following limits from first principles:

(i) $\lim_{x \to a} \dfrac{\sqrt{x} - \sqrt{a}}{x + a}$, 2

(ii) $\lim_{x \to a} \dfrac{\sqrt{x} - \sqrt{a}}{x - a}$, 3

(b) Calculate the derivative from first principles of the following functions at the indicated values of x:

(i) $f(x) = \sqrt{x}$ at $x = a$, 3

(ii) $f(x) = \frac{1}{\sqrt{x}}$ at $x = b$, 4

(c) Prove the Chain Rule: $(f \circ g)'(a) = f'(g(a)).g'(a)$, where $(f \circ g)(x) \equiv f(g(x))$. 10

(d) Differentiate the function f, where 3

$$f(x) = \sin(\cos(\sin x))$$

5 (a) Find (if they exist) the: 12

- roots
- critical points (maximum/minimum points & points of inflection)
- vertical asymptotes
- horizontal/diagonal asymptotes

of the function $\frac{x^2+x-1}{x-1}$

(b) Sketch the graph of f, using the information assembled in part (a). 13

6 In this question you will use a simple application of differential calculus to derive an important result from Optics, **Snell's Law**. The Figure shows a 25

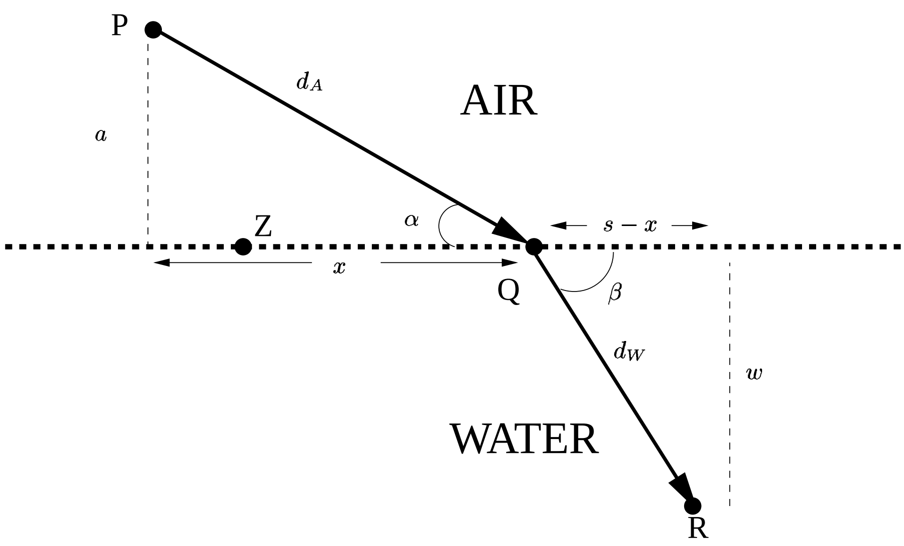

Figure 1: Deflected Light Ray

ray of light travelling from a point P at the top left of the diagram to R at the bottom right. The dotted horizontal line represents the boundary between air above the line and water below the line. Light travels at a speed c in air and at a slower speed v in water. Notice that the ray changes direction at the air-water boundary.

There is no obvious path for the ray of light to take between P and R. It could for example take the path P–Z–R or the path P–Q–R. To solve the problem we take from Physics the principle that:

"a ray of light passing between two points takes the path which **minimises** the time required". Referring to the diagram,

- a and w are constants; the vertical distance travelled by the light ray from P to Q (in Air) and from Q to R (in Water), respectively.
- s is a constant; the horizontal distance travelled by the light ray from P to R.
- x is a variable; the horizontal distance travelled by the light ray from P to Q (before hitting the Air/Water boundary).
- d_A and d_W are the distances travelled by the light ray in Air & in Water respectively.

(a) Find a formula for the distances d_A and d_W — defined above — in terms of x.

(b) Find a formula for the time T_A taken for the ray of light to travel from P to Q at speed c in terms of x — remember that $\text{Time} = \dfrac{\text{Distance}}{\text{Speed}}$.

(c) Find a formula for the time T_W taken for the ray of light to travel from Q to R at speed v in terms of x.

(d) Writing $T = T_A + T_W$, find the derivative of T with respect to x.

(e) Show that requiring that $\frac{dT}{dx} = 0$ implies that

$$\frac{\cos\alpha}{\cos\beta} = \frac{c}{v}. \quad \textbf{Snell's Law}.$$

N.B. you do not need to solve for x.